Introduction to Spin Networks and Towards a Generalization of the Decomposition Theorem

Hans-Christian Ruiz

8th March 2013

Acknowledgements

I am truly indebted and thankful for the help and advice of Prof. John Barrett; without him this work could never have existed. I would also like to recognize the valuable help and support of the staff at the University of Nottingham and the University of Munich (LMU). I am sincerely and heartily grateful to both my student advisor Bernhard Emmer and the Erasmus Institutional Coordinator of the LMU Jean Schleiss. I am sure this work would not have been possible without their support. Thanks are also due to Dr. Hal Haggard for useful comments.

Contents

0.1 Introduction

The objective of this dissertation is twofold. On one hand, it is intended as a short introduction to spin networks and invariants of 3-manifolds. It covers the main areas needed to have a first understanding of the topics involved in the development of spin networks. The topics are describe in a detailed but not exhaustive manner and in order of their conceptual development such that the reader is able to use this work as a first reading. On the other hand, some results aiming towards a decomposition theorem for non-planar spin networks are presented in Chapter 4.

We start in Chapter 1 with the first conceptual development of spin networks by Penrose, [30, 31], and a very brief presentation of the main concepts in the theory, [26], which will then be explained in more detail throughout the dissertation. Section 1.1 gives the motivation for considering spin networks as a way of constructing a 3-dimensional Euclidean space, as well as its formal framework of Abstract Tensor Systems, which is a generalization of the concept of tensor algebra. In fact, the diagrammatical language of spin networks are a representation of such systems. The arguments given are then reinforced when the description of General Relativity without coordinates due to Regge is presented, [34], and the Ponzano-Regge theory connecting both concepts is described, [33]. In Section 1.2 we will see that the information about the curvature of an n-manifold is encoded in the $(n-2)$-skeleton of its triangulation and in Section 1.3 we will discuss the relation between the asymptotic formula for the $6j$-symbols and the path integral over the exponential of the Einstein-Hilbert action. This relation represents the possibility of a similar description to QFT of a "Feynman integral" over geometries of a combinatorial manifold.

In Chapter 2 we present the basic mathematical framework for the algebraic description of spin networks via quantum groups, Section 2.1 following [25]. These algebraic objects belong to the family of quasitriangular Hopf algebras and the one deserving our special attention is the q-deformed universal enveloping Hopf algebra $U_q(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ which gives the data needed to regularize the Ponzano-Regge theory encountered before. The fact that the set of representations of this quantum group is finite, allows the construction of a well-defined invariant of 3-manifolds. We will discuss the corresponding state sums given by Turaev and Viro in Chapter 3. In 2.2 we introduce category theory, [24], in order to understand spherical categories, [8, 7], and their relation to the diagrammatic representations of spin networks given by the Temperley-Lieb recoupling theory in Section 2.3 following [20, 19]. The main concept needed from spherical categories is theorem 25 on page 48, which states that, given an additive spherical category, it is always possible to build a quotient in order to construct a non-degenerate spherical category. This construction allows the condition of semisimplicity needed for the construction of the above mentioned invariants. Hence the concept of semisimple spherical categories given by Barrett generalizes the Turaev-Viro invariants given by the state sum of $U_q(\mathfrak{sl}_2)$, [39]. We give in Section 3.3 a brief account of this generalization.

On the other hand, the diagrammatic language of spin networks is contained in the very broad field of knot theory. We will give a very brief account of knot theory and refer the

reader to [19, 18, 36] for a more detailed description. The most important concept of this area for us is the Temperley-Lieb recoupling theory, which allows a definition of the n-edges in a spin network as a weighted sum over representations of the generators of the Artin braid group B_n. Furthermore, this theory is important as a tool for the evaluation of closed spin networks in terms of the q-deformed $6j$-symbols, especially in the case where q is a root of unity, since it delivers all relations related to Moussouris' algorithm and the correspondence of a tetrahedron with the recoupling coefficients, relevant to the Ponzano-Regge partition function.

In Section 2.4 we present briefly the axioms of a topological quantum field theory (TQFT) related to the concepts mentioned above, [4, 5, 39]. The main idea needed in the description of the invariants of 3-manifolds, in fact of spin networks as well, is the association of finitely generated modules over some ring to (oriented) closed smooth manifolds of a fixed dimension d. In addition, to any $(d+1)$-cobordism between two such manifolds one associates an element of the module associated to its boundary, which is usually a tensor product of modules associated to each component of the boundary. The presentation of TQFT here is intended merely as a support to describe the Turaev-Viro invariants and, regrettably, it was not possible to dedicate more space to this topic, which could be a (very interesting) dissertation on itself. For a detailed discussion of the relation between link and knot theory and TQFT we refer to [36]. Section 2.4 is directly connected to Section 3.1, where a TQFT arises naturally from the Turaev-Viro state sum after the construction of a suitable quotient defining a functor from the category of cobordisms of triangulated 2-manifolds to the category of modules over a ring introduced for the initial data, [39]. The data for the construction of the state sum is introduced in a general setting. After discussing the topological aspects of the theory in Section 3.2, such as the transformations on triangulations called Alexander moves, [40], and their dual form called the Matveev-Piergallini moves, we conclude with the correspondence of the Turaev-Viro and the Kauffman-Lins invariants. We give this correspondence in an informal manner based on the comparison of the results in [39] and [20]. For a formal discussion the reader is referred to [32].

In Chapter 4 we carry out the original research towards a decomposition theorem for non-planar spin networks. We start by presenting some basic concepts of (topological) graph theory, mainly Kuratowski's Theorem 41 on page 80, in order to prepare the setting in which we will work, [9, 16]. The main idea in Section 4.1 is to identify the graphs corresponding to non-planar spin networks and construct the surfaces in which they are *cellular* embeddable by applying the so called rotation rule. It turns out that the only graph that needs to be studied is the $(3,3)$-bipartite graph $K_{3,3}$ since the other subgraph responsible for the non-planarity of a given spin network is the complete graph on 5 vertices, denoted K_5. This graph is expandable to the Petersen graph which has as one of its minors $K_{3,3}$. We used exclusively cellular embeddings to assure that the information about the topology of the surface is encoded in the graph itself and found readily that, for each graph, there is a topological constraint reducing strongly its possible cellular embeddings. Some examples for $K_{3,3}$ were calculated from the Rotation Scheme Theorem 42 on page 82 and it was proven that these are the only possible ones up to permutation of the vertices. In Section 4.2 we present Moussouris' algorithm for the evaluation of planar spin networks, [28], and extend it to account for the phase factors containing the information of a toroidal spin network. In addition, the Decomposition Theorem 47 on page 89 is improved to account for this

simple non-planar case and its irreducible network, named toroidal phase factor, is given. An attempt to evaluate the $K_{3,3}$ spin network is made by associating the irreducible toroidal network to the twisting factor given in [20, 13]. This factor changes effectively the orientation of a vertex and was used (may be in a naive way) to turn a toroidal phase factor into a theta-net translating the diagrammatic information of the topology into an algebraic factor, leaving a possible evaluation in terms of q-$6j$-symbols and the twist factor; this evaluation was called toroidal symbol. The results of some calculations regarding the different embeddings are given and compared. This shows some ambiguity in the evaluations calculated, which are discussed and a solution is proposed. Finally, the further work needed to achieve a general theorem concerning the evaluation of non-planar spin networks of higher genus is described briefly. We hope that the simple categorization of orientable surfaces as a connected sum of tori is reflected in the evaluation of spin networks in this general case, namely, as a sum of products of toroidal symbols. The further analysis regarding this generalization and the possible solution to the ambiguity of the evaluations calculated will be presented soon in an article containing also the results presented in this dissertation.

For the sake of readability the author was careful to avoid as much as possible mathematical details, keeping in mind the necessity of further explanation in key aspects of the dissertation. In many cases, however, this was difficult to achieve without extending too much the scope of the dissertation. Hence, for a detailed description and further aspects of the different topics, as well as the proofs of the statements made, the reader is referred to the literature given.

Most diagrams were made with the "Xy-Pic" package and the help of Aaron Lauda's tutorial. For this and more documentation concerning this useful package we refer to the Xy-pic website (http://www.tug.org/applications/Xy-pic/).

Chapter 1

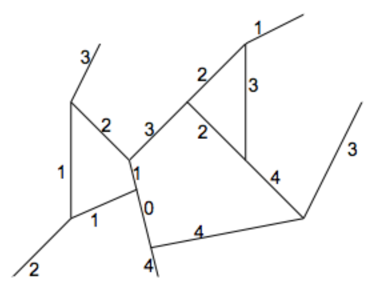
From Spin Networks To General Relativity

1.1 Spin Networks

1.1.1 The Origins of Spin Networks

Penrose constructed a discrete model of space based on the concept of quantum mechanical angular momentum with the goal to build a consistent model from which classical, continuous geometry emerged in a limit. It was shown that spin networks could reproduced a 3 dimensional Euclidean space, this result is known as the *spin-geometry theorem* (cf. Theorem 1). The basic idea of Penrose[1] was to build up space-time and quantum mechanics simultaneously from combinatorial principles by using as primary concepts the rules for combining angular momenta together. Continuous concepts, such as directions in space, should then emerge in a limit where the systems of angular momenta get more complicated.

In order to avoid referring to the idea of spins having $2s+1$ available states as preexisting directions of a background space, one has to work only with the total angular momentum (j-value) rather than the direction of quantization (m-value)[2]. Now imagine an object, called spin network, like the following

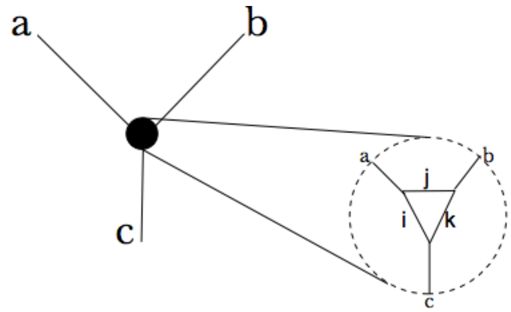


where the number next to each line represents twice the angular momentum of an isolated

[1]The following discussion can be found in [30].

[2]It is important to notice that a direction of quantization only appears when the system is related to a bigger one, e.g. a magnetic field, which defines a "preferred" direction.

and stationary subsystem[3], represented by the line and called an n-unit or n-edge. The above diagram is only a representation of the, for now, rather abstract concept of spin network, thus it has no spatial meaning, only a relational one, i.e. its defining properties are the relations between the edges[4]. Notice that an important element of this diagram is the trivalent vertex[5]



where the dashed circle indicates spin network structure at the vertex with *internal labels* i, j, k being positive integers determined by the external labels a, b, c

$$i = (a + c - b)/2, \; j = (b + c - a)/2, \; k = (a + b - c)/2.$$

The external labels must satisfy the triangle inequalities and add up to an even integer, these conditions are necessary as an expression of conservation of angular momenta.

Remark. The subsystems are not moving relative to one another, there is only a transfer of angular momentum allowed and the regrouping into different subsystems, not even time-ordering of the events play a role, only the topological aspects of the system are relevant.

Since only the relational properties of the edges define the spin network some combinatorial rules must be given, but how are this combinatorial rules to be interpreted? Every diagram will be assigned a non-negative integer called its norm, which can be calculated from any given spin network in a purely combinatorial way. We can think, carefully, of this as the measure of the frequency of occurrence of the given spin network, so we could use this norms in some cases to calculate the probabilities of different spin values occurring. Since the norm is always an integer these probabilities will always be rational numbers. How can these rules be obtained? A natural choice would be to derived them from irreducible representations of $SO(3)$.

How does this enable us to build up a concept of space? In other words, how can anybody say anything about directions in space, if there is only the non-directional concept of total angular momentum? One could ask for the *"orientation" of a n-unit in relation to some larger structure* belonging to the system under consideration. Hence, the system should involve a fairly large total angular momentum number N (called a large unit), in order to have the possibility of a well-defined direction as the spin axis of the system[6]. After defining this

[3]The system has to be regarded as isolated and stationary so it has a well-defined angular momentum, $j\hbar = \frac{1}{2}n\hbar$, $(n = 0, 1, 2, \dots)$.

[4]From this point of view, we can consider a spin network as a (cubic) graph.

[5]The number of edges at a vertex is not limited to three, e.g. in quantum gravity spin networks are generalized to include higher valence vertices, as pointed out in [26].

[6]Recall that, because of the correspondence principle, systems with large quantum numbers behave nearly as classical ones.

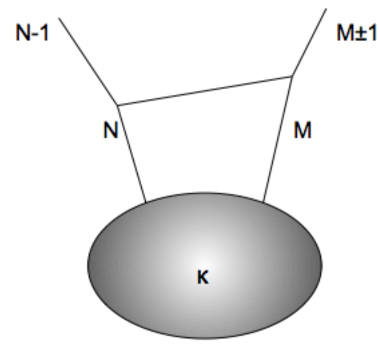
"direction", one can then ask further how the concept of angles between these "directions" could be defined, and then prove if we get a consistent interpretation of this in terms of directions in a 3-dimensional Euclidean space.

To define an angle between two large units Penrose considered following "experiment". Suppose a 1-unit is detached from a large N-unit such that it leaves an $(N-1)$-unit behind and it is then re-attached to some other large M-unit. According to the rules allowed, which are described below, we have then either an $(M-1)$-unit or an $(M+1)$-unit, thus there will be certain probabilities for these two different outcomes. With the information given by these probability values we can obtain the angle between the N-unit and the M-unit. To see how this is possible, notice that if the units are "parallel", then we would expect zero probability for the $M-1$ value and if the units are anti-parallel we would expect zero probability for the $M+1$ value. For a "perpendicular" position, we expect then equal probability values for each of the two outcomes. Hence, for an angle θ between the directions of the two large units we would expect a probability $P\left(M\pm 1\right)=\frac{1}{2}\pm\frac{1}{2}\cos\theta$ for the M-unit to be increased/reduced one unit. In this way, from knowledge of a spin network, one can calculate by means of a combinatorial procedure the probability of each of the two possible outcomes and with them define an angle between these two large systems. Since these type of probabilities will always be rational numbers, one could only obtain angles with rational cosines, but these "angles" would normally not agree with the actual angles of the Euclidean space until one goes to the limit of large systems. This means that rational probabilities, e.g. $p=m/n$, can be seen as something more fundamental than ordinary real number probabilities. The former might be regarded as arising because nature has to make a choice between m alternative possibilities of one kind and n alternative ones of another, all of which are to be *equally probable.* Only in the limit then, when numbers get to infinity, we would get the full continuous spectrum of probability values.

Now, consider a number of disconnected systems, each of them producing a large N-unit. To measure the "angle" between two of them use the above experiment. Then, the probability (cf. (1.2)) of the second N-unit to become an $(N\pm 1)$-unit is $\frac{1}{2}\left(N+1\pm 1\right)/\left(N+1\right)$. Hence, the probabilities become equal in the limit $N\to\infty$ and so, for large N, we could assign a right-angle between these two units. With this result one could put any number of N-units at right-angles to each other, thus, there is no restriction of the dimensionality of our space.

Clearly something went wrong. Remember there were no connections between any of the N-units so there is absolutely *no information* concerning the interconnections between the different N-units (there is a lack of knowledge concerning the origins of the systems), so we can think of the probability, in this particular case, as arising *entirely* out of the ignorance of the system, rather than being genuine quantum mechanical probabilities due to an "angle".

Suppose there is some "known" connecting network κ with two large units coming out, and we realize the above experiment two times consecutively.



If this is an "ignorance" situation, i.e. we do not know about how the spin axes are pointing, then the probabilities in the second experiment will change by the information of the relative orientation of the spin axes obtained in the first one. Hence, if the probabilities calculated for the second experiment are substantially altered, then there is a large "ignorance" factor involved. On the other hand, if they are not substantially altered, then the angle between the two large units is well-defined.

Suppose we have now a system which has many large units emerging $(A, B, ...)$ and that the angle between any two of them is well-defined. Then, without proof[7],

Theorem 1. Spin-geometry Theorem: *In the situation above, all angles between the large units are consistent with angles between the directions in a three-dimensional Euclidean space.*

Does this space corresponds to the original, given three-dimensional Euclidean space in the sense of a background? Well, there can be systems (n-units) with a large total angular momentum, but which do not give well-defined directions in the original space, remember, there are states with a large angular momentum which point all over the place (e.g. $m = 0$ states). Thus, the angles coming form these units do not correspond to anything one can see as angles in the original space, but they are nevertheless consistent with the angles between directions in some abstract Euclidean 3-space. One could take the view that the Euclidean three-dimensional space that comes out of the large units of spin networks is the *real* space, and that the original space is just a convenience, like coordinates in general relativity. The main idea here is the claim that *the system defines the geometry.*

In order to define the probabilities of a given process, the definition of the norm of a spin network is needed. This will be given in terms of a concept called the **value of a closed spin network**. A spin network is closed if it has no free ends. If it is not closed a value will not be assigned, but one can always obtain a closed network by making a copy of the "open" one and gluing the corresponding edges together. To define the value of a closed spin network, replace each n-unit in the diagram by n parallel strands. At each vertex, the strands belonging to different edges must be connected together in pairs, such that two strands of the same unit are not allowed to be connected (cf. Remark 6). Such a connection is called a *vertex connection.* The *sign of a vertex connection* v is defined as $s_v = (-1)^x$ where x is the number of intersection points between different strands at the vertex[8]. When the

[7] For proof see [28].

[8] This can also be seen as the signum of the permutation of strands involved in their pairing at each vertex connection, cf. Definition 5.

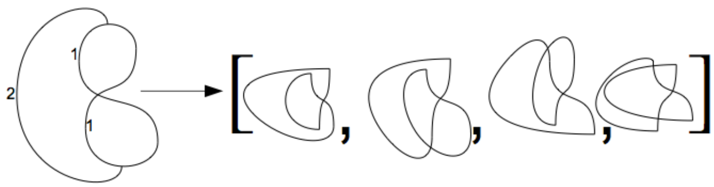
vertex connections have been completed at every vertex of a closed spin network, then we get a number c of closed loops. The value of the closed spin network is the sum over every possible way of completing the vertex connections. From these considerations, Penrose got the following expression

$$value = \frac{\sum s\,(-2)^c}{\prod n!}, \tag{1.1}$$

where $s = \prod_v s_v$ and $\prod n!$ ranges over all the n-units of the spin network.

Remark. (i) In the calculation of the value only the intersections at the vertex connections count for the sign of the diagram, see example below. *(ii)* Note that each closed loop yield a value of -2 to the main diagram. *(iii)* The value of any closed spin network always turns out to be an integer.

Example 2. Consider the following diagram and its possible vertex connections



Note that the intersection arising from the crossing of the two 1-units does not contribute to the sign of the terms in the sum. The evaluation of the above diagram according to (1.1) gives

$$value = \frac{1}{2!1!1!}\{+(-2) - (-2)^2 - (-2)^2 + (-2)\} = -6.$$

The value of a spin network is multiplicative, i.e. the value of the union of disjoint spin networks is equal to the product of their individual values. Before coming to the definition of the norm we remark the existence of some useful reduction formula for evaluating spin networks which may be substituted into any closed spin network in order to obtain a valid relation between values; they can be found in [30].

Definition 3. The **norm** of a given spin network is obtained by joining together the corresponding free end units of two copies of the diagram to make a closed network and taking the modulus of its value. In other words, the norm of a spin network is the **modulus of the value** of its corresponding closed spin network.

Finally, the norm is used to calculate the probabilities for spin-numbers in the sense of the above described "experiment". Given a spin network α with a free $a-$ and $b-$unit, suppose these two units come together to form an x-unit in a resulting spin network β. What are the various probabilities for the different possible values of x? Let γ denote the vertex formed by a, b and x and ξ denote the spin-network consisting of the x-unit alone. The probability for the resulting spin-number to be x is

$$probability(x) = \frac{norm\beta}{norm\alpha}\frac{norm\xi(x)}{norm\gamma(x)}. \tag{1.2}$$

With the interpretation of an "angle" given by Penrose the three-dimensional Euclidean nature of the "directions in space" is a consequence of the combinatorial probabilities of spin networks. One should keep in mind that this space is the one defined by the system and there *is* a distinction between this one and the space introduced as a background in a conventional formalism.

1.1.2 A Formal Framework for Spin Networks: Abstract Tensor Systems

In [31] Penrose describes his theory of *abstract tensor systems* (ATS) for more general objects than ordinary tensors and which are related to diagrams like the ones above. These objects are denoted formally identically as in the tensor index notation, but the meaning of indices is now different. Indices are now just a label, they do not stand for $1, 2, 3, ..., n$, and an element ξ^a of an ATS is not a set of components of a vector but rather a whole element of a module $\mathcal{T}^a$ over a ring $\mathcal{T}$. This means that $\xi^a \neq \xi^b$ if $a \neq b$ but $\mathcal{T}^a \cong \mathcal{T}^b$. This is naturally extended to other objects like $\chi^{ab...d}_{f...h} \in \mathcal{T}^{ab...d}_{f...h}$ where $a, b, ..., d, f, ..., h$ are all distinct and $\mathcal{T}^{ab...d}_{f...h}$ is a module over $\mathcal{T}$ [9]. The ATS has four basic operations which are

1. Addition: $\mathcal{T}^{x...z}_{u...w} \times \mathcal{T}^{x...z}_{u...w} \to \mathcal{T}^{x...z}_{u...w}$
2. Outer multiplication: $\mathcal{T}^{a...d}_{p...r} \times \mathcal{T}^{x...z}_{u...w} \to \mathcal{T}^{a...dx...z}_{p...ru...w}$
3. Contraction[10]: $\mathcal{T}^{px...z}_{qu...w} \to \mathcal{T}^{x...z}_{u...w}$
4. Index substitution: $\mathcal{T}^{x...z}_{u...w} \to \mathcal{T}^{f...h}_{k...m}$ where there is a one-to-one correspondence between both set of indices.

The axioms for these operations are the following

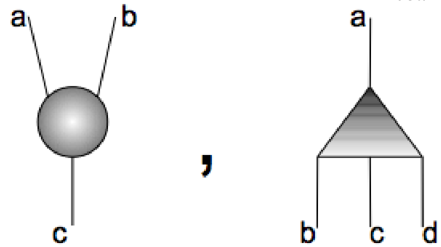
1. The addition defines an Abelian group structure for each module in the ATS.
2. The multiplication is distributive over the addition.
3. The contraction is distributive over addition and commutes with multiplication and other contractions.
4. The index substitution is caused by any permutation of the set of indices but the validity of any formula stays unaltered.

These abstract tensor systems may be expressed diagrammatically, what allows us to see connections between indices at a glance. Penrose pointed out, that if we regard the labels as points on a plane, we may denote an object of the ATS by a symbol with "arms" corresponding the upper indices and "legs" corresponding the lower ones.

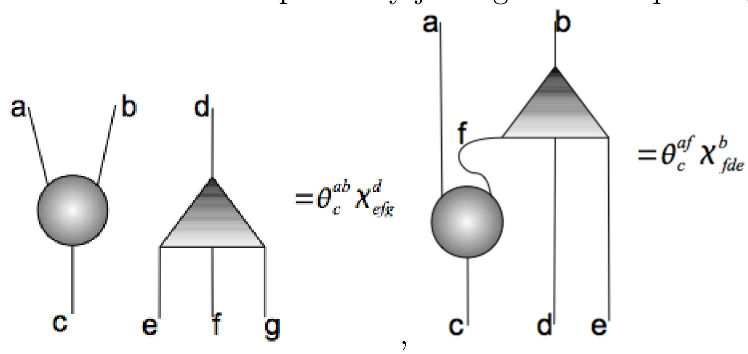
[9]The order of the indices of the objects is important, $\chi^{ab...d}_{f...h} \neq \chi^{db...a}_{f...h}$, but not for the modules $\mathcal{T}^{ab...d}_{f...h} = \mathcal{T}^{db...a}_{f...h}$.

[10]We will use the dummy index notation and the Einstein's sum convention.

Example 4. If $\theta_c^{ab} \in \mathcal{T}_c^{ab}$ and $\chi_{bcd}^{a} \in \mathcal{T}_{bcd}^{a}$ we denote these object by



respectively. Now, outer products are expressed as a juxtaposition of individual symbols and contractions are depicted by joining the corresponding arm and leg:

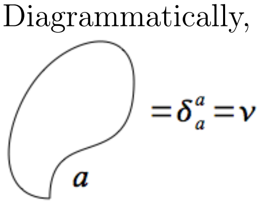


The addition of two objects is analogous. One simply draws the diagrams for each object and puts a "+" sing in-between.

We require the ATS to have "unit elements" $1 \in \mathcal{T}$ such that $1\chi_{\cdots}^{\cdots} = \chi_{\cdots}^{\cdots}$ for all elements of $\mathcal{T}_{\cdots}^{\cdots}$ and $\delta_b^a \in \mathcal{T}_b^a$ such that $\chi_{\cdots}^{\cdot\cdot p \cdot\cdot}\delta_p^q = \chi_{\cdots}^{\cdot\cdot q \cdot\cdot}$ and $\chi_{\cdot\cdot x \cdot\cdot}^{\cdots}\delta_y^x = \chi_{\cdot\cdot y \cdot\cdot}^{\cdots}$. These elements are unique and the element δ_b^a has the formal properties of a Kronecker delta, so the definition of the "dimension" of $\mathcal{T}^a$ arises naturally to be the scalar $\delta_a^a = \nu$.

Remark. In ordinary tensor systems ν is a positive integer, while in the more general case presented here it could also be a negative integer, see remark 6.

Diagrammatically, the "dimension" is depicted as a closed loop as the following



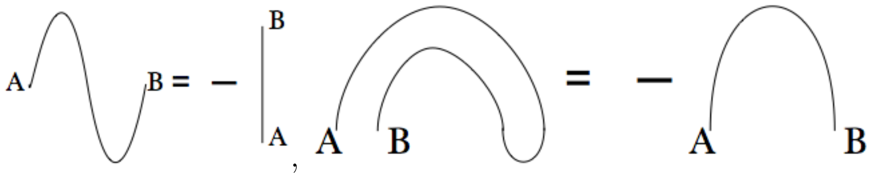
This is the general framework for the discussion about spin networks in the beginning of this section. Now, we can re-introduce the spin networks by associating 2×2 matrices with diagrams, as in [26]. However, we have to make certain that the diagrammatics are planar isotopic, i.e. invariant under smooth deformations in the plane, cf. Section 2.3. From the considerations above, we start by associating[11] the Kronecker delta symbol δ_A^B, which in this case is the 2×2 identity matrix, to a line:

$$\delta_A^B \to \Big|_A^B$$

[11] This association implicitly fixes a preferred direction from the bottom to the top of the page, i.e. $\big|_A^B \leftrightarrow A \text{---} B$.

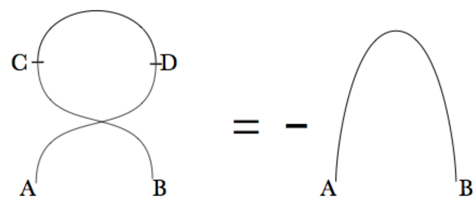
Another possible identification[12] is between a curve and the antisymmetric matrix $\varepsilon_{AB} = \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; i. e. $\varepsilon_{AB} \to_A \bigcap_B$ and $\varepsilon^{AB} \to^A \bigcup^B$.

These definitions are, however, not planar isotopic since the identities[13] $\delta^C_A \varepsilon_{CD} \varepsilon^{DE} \delta^B_E = \varepsilon_{AD}\varepsilon^{DB} = -\delta^B_A$ and $\varepsilon_{AD}\varepsilon_{BC}\varepsilon^{CD} = -\varepsilon_{AB}$ correspond to the following diagrams



As pointed out in [26], in order to solve this problem we simply re-define the bent line according to $\varepsilon_{AB} \to \tilde{\varepsilon}_{AB} = i\varepsilon_{AB}$.

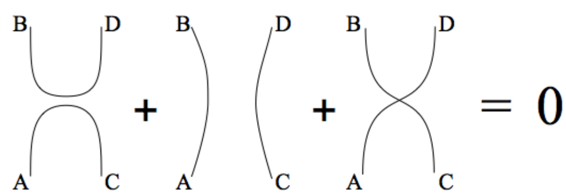
Now, consider the relation $\delta^D_A \delta^C_B \tilde{\varepsilon}_{CD} = -\tilde{\varepsilon}_{AB}$, which gives the diagram



Such difficulty can be solved by associating a minus sign to each crossing. All these modifications ensure the planar isotopy and as explained above gives us the possibility to perform algebraic calculations in a more transparent way (cf. Example 4). For instance, the value of a simple closed loop is negative, since $\tilde{\varepsilon}_{AB}\tilde{\varepsilon}^{BA} = -\varepsilon_{AB}\varepsilon^{BA} = -2$.

There is an important relation called the skein relation (or spinor identity) which is derived from a known identity concerning the product of two antisymmetric tensors[14]:

$$\epsilon_{AC}\epsilon^{BD} = \delta^B_A \delta^D_C - \delta^D_A \delta^B_C \Leftrightarrow$$



This algebra is "topological", i.e. any two diagrams which are homotopic to each other represent two equivalent algebraic expressions. The spin network diagrammatics are topologically invariant in a plane as a consequence of a result of Reidemeister: a knot, i.e. an embedding of S^1 in a three dimensional Euclidean space, is homotopic to another knot, if and only if, the planar projection of the knots can be transformed into each other via a finite sequence of Reidemeister moves[15]:

[12] Again, the association implies a direction from the first index to the second which has to be kept in mind when evaluating closed networks and notice that the orientation has to be consistent throughout the diagram.

[13] This identities are associated to the more general notion of a pivotal category, cf. Def. 20 on page 43.

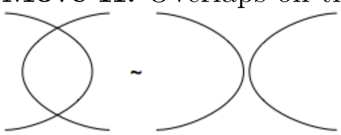
[14] Note that for the diagrammatic relation the sign of the crossing changes.

[15] In three dimensions one has different types of crossing, the "over crossing" and the "under-cross". Here we just depicted the projections on the plane so, in the general case, on each intersection one has to keep in mind the additional structure of the crossing, cf. Section 2.3.

- **Move 0:** In the plane of projection, one can deform the following curve smoothly

- **Move I:** A curl may be undone since we are dealing with one dimensional objects

- **Move II:** Overlaps on the projection plane of distinct curves are not knotted

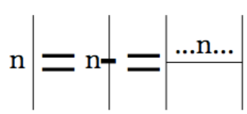


- **Move III:** Planar deformations under or over a diagram

With a finite number of these moves the projection of a knot may be transformed into the projection of any other knot which is topologically equivalent to the first one, this equivalence is called ambient isotopy. Planar isotopy is a special case of this, with the restriction that there are no crossings, only intersections, cf. Section 2.3.

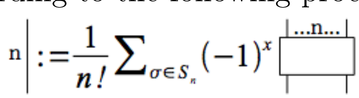
Finally, we give the definition of an edge as the one used by Penrose in the discussion above. This edge must be independent of all the possible linear relations the graphs might have, e.g. skein relations. Thus, for the n "strands" of an edge one sums over all permutations of the lines considering the sign of the permutation involved:

Definition 5. An **n-edge** (or n-unit) is a set of lines woven into a single graph denoted by



,

The weaving is according to the following procedure,



,

where x is the minimum number of transpositions needed to generate the corresponding permutation and the box with the n strands denotes their permutation σ, cf. Section 2.3.1.

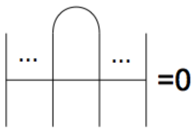
As pointed out in [31] this is exactly (in the case of ordinary tensors) the so-called *generalized Kronecker delta*

$$\begin{vmatrix} \delta^a_p & \delta^b_p & \dots & \delta^f_p \\ \delta^a_q & \delta^b_q & \dots & \delta^f_q \\ \vdots & \vdots & \ddots & \vdots \\ \delta^a_u & \delta^b_u & \dots & \delta^f_u \end{vmatrix} = \delta^{ab\dots f}_{pq\dots u}$$

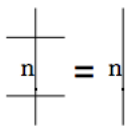
since each term in the definition above is an outer product of n "Kronecker deltas".

Remark 6. *(i)* The antisymmetrizer of the lines in the definition of the n-edge actually symmetrizes the indices in the $\delta\epsilon-$world since each crossing provides an extra sign to the corresponding term, [26]; *(ii)* The properties of the antisymmetrizer are the following, cf. Section 2.3.1:

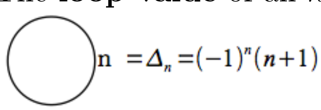
- **Irreducibility**, this means that an edge vanishes when a pair of lines is retracted:



- The antisymmetrizer act as **projectors**:



- The **loop value** of an n-edge is an integer[16]:



Now we can define a spin network as consisting of a graph with edges, vertices and labels. The labels represent the number of strands woven into edges and any vertex with more than three incident edges must also be labeled to specify a decomposition into trivalent vertices, cf. Section 2.3.2 and [35].

As the strands from which spin networks are woven can take two values, they are well-suited to represent two-state systems, hence the name "spin network". Thus, it is possible to include the $|jm\rangle$ representation of angular momentum into the diagrammatics of spin networks, where each n-edge corresponds to the $\frac{n}{2}$-irreducible representation of SU(2), hence all results of representation theory have a diagrammatic form. For a detailed description of the angular momentum representation consult [26]. In fact, as described above, this specific representation is only a special case from a much richer and general theory involving representations of other algebras which will be explained in the next chapter, but first it is important to explore the motivation to consider these abstract objects in the first place. We will do so by considering general relativity in the framework of the Ponzano-Regge theory.

1.2 General Relativity

In 1961 T. Regge described an approach to the theory of Riemannian manifolds that enable the description of general relativity without the use of coordinates, [34]. Following the original work of T. Regge in this section we describe the main ideas of his approach.

[16] The multiplicity $(n+1)$ follows from the recursion relation $\Delta_{n+2} = (-2)\Delta_{n+1} - \Delta_n$; $\Delta_0 = 1$, $\Delta_1 = -2$, cf. Sec. 2.3.1.

As it is generally known, one can define an intrinsic Gaussian curvature on any surface by carrying out measurements on geodesic triangles, [27, p. 336f]. This is due to the fact that, if the triangle T has non Euclidean geometry, in general $\alpha+\beta+\gamma\neq\pi$, where α,β,γ are the internal angles of T. From this, one may define the Gaussian curvature ϵ_T of T by $\epsilon_T=\alpha+\beta+\gamma-\pi$.

To get a relation between the area of a spherical triangle T and the radius R of the sphere consider the area of a segment S_α of a sphere. If this segment is built from two arc segments, i.e. segments of two great circles, with an angle α between them, leaving both from the point A on the sphere and intersecting in the opposite point A', then $S_\alpha=2\alpha R^2$. Now consider three great circles forming T by intersecting at the points A,B,C and dividing the surface of the sphere in different segments. With the above relation for each segment of the sphere one obtains the following relation between the radius R, the area A_T of the triangle T and the Gaussian curvature ϵ_T (cf. [38]),

$$\epsilon_T=\frac{A_T}{R^2}.$$

Suppose that T shrinks to a point P and the limit $\frac{\epsilon_T}{A_T}\underset{A\to 0}{\to}K(P)$ exists independent of the limiting procedure, then we can take

$$K(P)=\underset{A\to 0}{lim}\frac{\epsilon_T}{A_T} \tag{1.3}$$

as the definition of (local) Gaussian curvature at the point P.

Consider a polyhedron M and suppose the triangle T lies in the interior of a face of M or an edge of M crosses it. Then $\epsilon_T=0$ since in the former case T is flat and in the latter the neighborhood of a point in an edge is homeomorphic to a neighborhood in the plane. Therefore $K(P)=0$ if P is not a vertex. On the other hand, if T contains a vertex V the Gaussian curvature is independent of the form of T and $\epsilon_T=\epsilon_V$ is a constant obtained from the relation

$$\epsilon_V=2\pi-\sum_i\alpha_i \tag{1.4}$$

where the sum is over the angles α_i at the vertex V. If T contains several vertices, then we have

$$\epsilon_T=\sum_V\epsilon_V$$

so if we think of $K(P)$ as a Dirac distribution with the vertices V as support, i.e. $K(P)\sim\sum_V\delta(P-V)$, we can use the original formula for the Gaussian curvature $\epsilon_T=\int_T K(P)dA$. Thus, from the above, we have a relation between the curvature of a polyhedron and the deficiency of its vertices, [34].

Example. The simplest triangulation of S^2 is a (regular) tetrahedron. Hence, the deficiency of each vertex is $\epsilon_V=2\pi-(\alpha_1+\alpha_2+\alpha_3)=\pi$. This gives an Gaussian curvature for the tetrahedron of 4π. Compare this result with the integral curvature of a sphere with radius R and $K=1/R^2$, which is $\epsilon_S=\int_S KdA=4\pi$.

Now, the Gauss-Bonnet theorem for a closed surface[17] links the curvature of a manifold M with the Euler characteristic $\chi(M) = 2 - 2g$, where g is the genus of M,

$$\int_M KdA = 2\pi\chi(M).$$

Hence, in the case of a polyhedron, the deficiency angles at the vertices and the Euler characteristic are connected such that the Poincaré-Euler formula is obtained directly by first carrying out the sum over all faces f having V as a common vertex (cf. (1.4)) following a summation over all vertices and then the other way around, first vertices and then faces. Since all faces of M are triangles[18], we obtain

$$V - E + F = 2 - 2g$$

where V is the number of the vertices, F the number of faces and E the number of edges.

After this small deviation showing the tight relation between geometry and topology, consider a triangulation of a manifold M. Knowing all lengths of all edges and the connection matrix[19] gives us enough information to know also all internal angles α_{fn} of the face f with vertex V_n. Thus we know all deficiencies ϵ_n at all vertices and therefore the intrinsic curvature of M. Notice that whenever the number of vertices (and with it the number of edges and faces) increases the local Gaussian curvature approximates to a continuous function of the density of vertices ρ and their deficiency ϵ, $K(P) = \rho\epsilon$, when the variation of the product in the triangle T around the point P is small, cf. Eq. (1.3).

In the case of higher dimensional cell complexes the notion of a geodesic triangle has to be replaced by the notion of parallel transport, i.e. an orthogonal mapping between the tangent space $T_P M$ at P, and the tangent space $T_Q M$ at another point Q, whenever the points P and Q are connected by a path a in M. If a is a loop, $T_P M$ is mapped to itself, the mapping being a rotation around P by an angle $\epsilon(a)$. Since the rotation is proportional to the curvature (cf. [29, Sec. 7.3.2]), we obtain

$$\epsilon(a) = \int_a KdA.$$

This is an additive function of the loops, i.e. if a, b are loops, then $\epsilon(a \bullet b) = \epsilon(a) + \epsilon(b)$, where $\bullet$ denotes the product of loops.[20] Whenever a vertex lies on the curve the parallel transport is not defined unambiguously, thus we consider the homotopy equivalence only in M/V where V denotes the set of all vertices in M so the fundamental group of M/V, denoted by $\pi_1(M/V)$, is not trivial if $V \neq \emptyset$.

---

[17]By closed surface it is meant a compact two-dimensional manifold without boundary.

[18]From this follows that the sum of angles over vertices V and faces F, $\sum_{F,V} \alpha_{FV} = \pi F$. Furthermore, we have the relation $2E = 3F$ between the number of edges and faces of a polyhedron.

[19]The connection matrix contains the information about the relation between faces, edges and vertices. Hence it corresponds to a metric in the continuous case.

[20]This product is defined as following:
Let M be a topological space. For each two loops $a, b : [0, 1] \to M$ at a point P with $a(1) = b(0)$, $a \bullet b(t) = \begin{cases} a(2t) \; 0 \leq t \leq \frac{1}{2} & b(2t-1) \; \frac{1}{2} \leq t \leq 1 \end{cases}$ where $t \in [0, 1]$. This product is naturally extended to the homotopy class of the loops in M such that, together with the homotopy class of inverse loops defined as $a^{-1}(s) \equiv a(1-s)$, $s \in [0, 1]$ for all $a : [0, 1] \to M$ and the unit element as the homotopy class of loops homotopic to P, it defines a group called the *fundamental group*, [29].

Now, each loop at P is associated to an orthogonal matrix $S(a)$ acting on $T_P M$. Since a loop homotopic to P is the boundary of a simply connected region in M/V, there is no vertex in that region of M so the curvature vanishes there. Hence the unit element $[u] \in \pi_1(M/V)$ represents the identity matrix in the association made before. On the other hand, consider a loop $c = a \bullet b$ which is not homotopic to P. Then the vector $x \in T_P M$ will be parallel transported first along a then along b, such that the resulting rotation of x is associated to a matrix $S(c) = S(b)S(a)$, where $S(c)$, $S(b)$, $S(a)$ are the matrices associated to the loops c, b, a respectively. In fact, the association is not between loops and orthogonal matrices, but rather the homotopy classes, i.e. the elements of $\pi_1(M/V)$, are represented by orthogonal matrices; to see this, we use the fact that if c, c' are loops in the same class, then $c' = v \bullet c$ where $v \in [u]$. Thus, with the previous, we have the relation $S(c') = S(c)S(v) = S(c)$, [34].

As an example, consider a one-sheeted cone, which is a polyhedron with one vertex only. It can be parametrized similar to a plane by polar coordinates (ρ, θ) such that the metric is given by $ds^2 = d\rho^2 + \rho^2 d\theta^2$, but with the identification of points with the same ρ and angles θ differing by a multiple of $2\pi - \epsilon$, where ϵ is the deficiency of the vertex $\rho = 0$. This manifold is called the $\epsilon-$cone and it can be extended to higher dimensional spaces by considering the product $\mathbb{R}^{n-2} \times \epsilon - cone$ with the metric $ds^2 = d\bar{z}^2 + d\rho^2 + \rho^2 d\theta^2$ where $\bar{z} \in \mathbb{R}^{n-2}$. This manifold, called the $\epsilon - n - cone$, is Euclidean everywhere but the $n-2$ dimensional flat subset $\rho = 0$.

Now, consider a loop in the $\epsilon - 3-$cone around the line $\rho = 0$, called the **bone**, with $\theta(0) = 0$ and $\theta(1) = N(2\pi - \epsilon)$, where N gives the winding number. Since only loops that are completed around the bone are not null homotopic, i.e. homotopic to a point, loops with the same N belong to the same homotopy class. Let the homotopy class be denoted by $a(N)$, then the multiplication of two equivalent classes is given by $a(N) \bullet a(M) = a(N+M)$, so the fundamental group is isomorphic to $(\mathbb{Z}, +)$. The orthogonal matrix associated to N is found by considering the fact that $S(0) = 1_{n\times n}$ and $S(M)S(N) = S(N+M)$ must hold, thus we can write $S(N) = S^N$ where S is called the generator of the bone. Let $V \in T_P \mathbb{R}^{n-2} \times \epsilon - cone$ be a vector in P, which can be split into orthogonal components $V^{||}$, $V^{\perp}$ lying in the subspaces $\mathbb{R}^{n-2}$ and $\epsilon - cone$ respectively. If we take V along $a(1)$, the component $V^{||}$ will remain unaffected by the process, i.e. $S(1)V^{||} = V^{||}$, so the subspace $\mathbb{R}^{n-2}$ is invariant under the action of the orthogonal $n \times n-$matrix $S(N)$. However, the component $V^{\perp}$ will be rotated by the angle ϵ. Hence, the general form $S(N)$ is a $n \times n-$matrix with two block-matrices in the diagonal, the first one is the identity matrix $1_{(n-2)\times(n-2)}$, and the second one is a rotation matrix describing a rotation on the $\epsilon - cone$ by the angle $N\epsilon$. In the particular case of an $\epsilon - n-$cone the fundamental group and its associated transformation group of orthogonal $n \times n-$matrices are Abelian, [34].

The $\epsilon - n-$cone described above is an example of n-dimensional generalizations of polyhedra, called skeleton spaces, where the curvature of the manifold M is a consequence of a $(n-2)-$dimensional subset $w \subset M$, called the skeleton of M. For the definition of these generalizations we start from a simplectic decomposition of the n-dimensional manifold. This determines the topology of this space but not the metric, which has to be defined.

Definition 7. As defined by Regge in [34], the (Euclidean) metric in M is given by the following axioms:

1. The metric in the interior of any n-dimensional closed simplex T_n is Euclidean[21].

2. In the metric of T_n, its boundary is decomposable into the sum of $n+1$ closed simplexes T_{n-1}, which are flat.

3. If a simplex T_{n-1} is common boundary of T_n and T'_n, the distance of any two points of T_{n-1} is the same in both frames of T_n and T'_n.[22]

4. If $P \in T_n,\ P' \in T'_n$ and $P,\ P'$ are "close enough" to T_{n-1} we define the distance PP' as $d(P,\ P') = \inf_{Q \in T_{n-1}}\{PQ + QP'\}$.

On the interior of T_{n-1} the metric is well defined and Euclidean as well. However, it might not be the case on the T_{n-2}'s of the boundary of T_{n-1}. The problem with the metric on T_{n-2} is that, with these axioms, it is not possible to define a metric since starting with the metric on a T_{n-1} that is common boundary of some n-simplexes does not ensure that at its boundary, i.e. at the T_{n-2} which is common boundary of *other* T_{n-1}'s, the metric will continue to be well defined, it could be that the metric defined in other adjacent T_{n-1} is different such that the metric would not be unambiguous.

This definition is enough to join smoothly neighboring simplexes so that one can construct a manifold which is everywhere Euclidean in M/w. In the 3-dimensional case the bones are straight segments connecting two 0-simplexes, so locally the bone is an $\epsilon - 3-$cone. If the lengths of these bones are chosen at random, the sum of all dihedral angles around the i-th bone will be $2\pi - \epsilon_i$.

Consider a 0-simplex T_0 in the above 3-dimensional manifold M_3 being the common end of several oriented bones $T_1^1, \ldots, T_1^m$. The orientation of the bones is such that all T_1^i's have, from T_0, outgoing direction; such a 0-simplex is called an oriented m-joint. Consider an idealized isolated m-joint where all the bones extend up to infinity and the set of plane sectors A_p formed by two contiguous bones $T_1^p,\ T_1^{p+1}$ have the property that $A_p \cap A_{p+1} = T_1^{p+1}\ \forall p \in \{1, \ldots,\ m\}$ and $A_i \cap A_j = \emptyset\ \forall i,\ j \in \{1, \ldots,\ m\}$, with $j \neq i+1$, where we identify $m+1 \triangleq 1$. Then $\mathcal{A} = \underset{p}{\cup} A_p$ divides the space M in two regions M' and M''. Suppose that $P \in M'$, $Q \in M''$ and that there are m paths t_p connecting P to Q. Let t_p be such that it intersects $\mathcal{A}$ only one time and only at A_p, therefore $a_p = t_p t_{p+1}^{-1}$ is a loop[23] around T_1^p. Notice that a_p encircles T_1^p only once, thus we have the identity $a_1 \bullet a_2 \bullet \ldots \bullet a_p = t_1 t_2^{-1} \bullet t_2 t_3^{-2} \bullet \ldots \bullet t_m t_{m+1}^{-1} = u$, which translate for the generators

$$S_1 S_2 \ldots S_m = 1_{3\times 3} \tag{1.5}$$

Example 8. Consider a 4-joint. Then we have the relation $S_1 S_2 S_3 S_4 = 1_{3\times 3}$, where S_i is the representation of the homotopy class corresponding to one loop around the i-th bone. Since

[21] This means that if one defines a coordinate system in T_n, one can also give the coordinates of the points of the boundary of T_n in this frame.

[22] Even if the coordinates might be different, the distance between two points in T_{n-1} is invariant under the change of coordinates from one simplex to the other.

[23] The definition of the product of two paths is the same as the one for the product of loops. Here we do not write explicitly the dot $\bullet$.

our manifold is three-dimensional we can write this representation as

$$S_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\epsilon_i) & -sin(\epsilon_i) \\ 0 & sin(\epsilon_i) & cos(\epsilon_i) \end{pmatrix}$$

where ϵ_i is the deficiency at the i-th bone. Carrying out the multiplication of the matrices we find in this case

$$\sum_{i=1}^{4} \epsilon_i = 2\pi n.$$

The relation (1.5) is the geometrical version of the Bianchi's identity encountered in a differential Riemannian manifold, as shown in [34]. In order to see this, Regge first shows the relation between the concepts discussed previously and the (coordinate expression of) Riemann curvature tensor by considering the transition from a skeleton space to a differentiable manifold, which is accomplished by increasing the density ρ of the bones but keeping the local curvature $\rho\epsilon$ slowly varying.

Consider a bundle of parallel bones on M_3 which induce a small curvature. Test the curvature of the manifold by carrying a vector V along a small loop with area[24] $\overrightarrow{\Sigma} = \Sigma\overrightarrow{n}$. Hence, the vector V would be rotated around a unit vector U parallel to the bones by the angle $\sigma = N\epsilon$, where N is the number of bones through Σ and it was assumed that each bone contribute the same deficiency ϵ. Assuming a uniform density ρ of bones and $\overrightarrow{\Sigma}||U$, we have $N = \rho(U, \overrightarrow{\Sigma}) = \rho\Sigma$, thus $\sigma = \rho\epsilon\Sigma$ or $\rho\epsilon\Sigma_\gamma U^\gamma$. Suppose the rotation is infinitesimal, such that in first order the rotated vector can be expressed as $V' = (1 + \sigma U\overrightarrow{S})V$ where S_κ are the generators of the rotation around U. Hence, for the change of the vector we obtain[25]

$$\begin{aligned} \Delta V &= \sigma(U\overrightarrow{S})V = \sigma U \wedge V, \\ \Delta V^\alpha &= (\rho\epsilon\Sigma_\kappa U^\kappa)(\epsilon^{\alpha\beta\gamma}U_\beta V_\gamma). \end{aligned}$$

Inserting $\delta^\tau_\kappa = \frac{1}{2}\epsilon^{\delta\omega\tau}\epsilon_{\delta\omega\kappa}$ and defining $U^{\alpha\gamma} = \epsilon^{\alpha\gamma\beta}U_\beta$ and $\Sigma^{\delta\omega} = \epsilon^{\delta\omega\tau}\Sigma_\tau$ we obtain

$$\Delta V^\alpha = \frac{1}{2}\rho\epsilon U^{\gamma\alpha}U_{\delta\omega}\Sigma^{\delta\omega}V_\gamma$$

which we compare with the standard form of the coordinate expression for the Riemann curvature tensor (cf. [29])

$$\Delta V^\mu = R^{\kappa\mu}{}_{\lambda\nu}\varepsilon^\lambda\delta^\nu V_\kappa; \text{ where } \varepsilon^\lambda\delta^\nu = \frac{1}{2}\epsilon^{\lambda\nu\tau}\Sigma_\tau.$$

Hence we obtain the relation for the approximation to differential manifolds given by Regge

$$R_{\mu\sigma\alpha\beta} = \rho\epsilon U_{\mu\sigma}U_{\alpha\beta}$$

[24]We change the notation at this point to emphasize that the area has a normal vector $\overrightarrow{n}$.

[25]With the following notation is intended to signalize that, even if the example is given for a 3-Manifold, the results can be generalized for higher dimensional cases.

which has all symmetry properties of the Riemann tensor, like the first Bianchi's identity $U_{\mu\sigma}U_{\alpha\beta} + U_{\mu\alpha}U_{\beta\sigma} + U_{\mu\beta}U_{\sigma\alpha} = 0$ and with $U_{\mu\sigma}U^{\mu\sigma} = 2$ we have that the scalar curvature $R = 2\rho\epsilon$ is independent of the dimension n of the space and twice the Gaussian curvature $K(P)$. Observe also that the orientation of a bone in M_n is determined by the skew symmetric tensor $U_{\mu\sigma}$, since in the case of a parallel bundle of bones, we are able to choose coordinates $\{x_i;\, i = 1, \dots, n\}$, such that x_1, x_2 are perpendicular and x_j $(j \geq 3)$ are parallel to the bone. Thus, $U_{12} = -U_{21} = 1$ while the other components of the tensor vanish.

Remark. (i) If the deficiencies ϵ_p are small, (1.5) can be written as

$$\sum_{p=1}^{m} \epsilon_p U^p_{\mu\sigma} = 0.$$

(ii) Assuming a distribution ρ of identical m-joints we can write the Riemann tensor as a sum over the m bundles of parallel bones, where the p-th bundle has defect ϵ_p, density ρ_p and direction U^p, as follows:

$$R_{\alpha\beta\lambda\delta} = \sum_{p=1}^{m} \epsilon_p \rho_p U^p_{\alpha\beta} U^p_{\lambda\delta}.$$

This leads to the correspondence between (1.5) and the Bianchi's identities, [34].

(iii) For a non-positive definite metric of a skeleton space, the generators of the bones are Lorentz matrices. In the case $n = 4$ the bones are triangles with area

$$4L^2 = (A_\mu B^\mu)^2 - (A_\mu A^\mu)(B_\mu B^\mu), \tag{1.6}$$

A_μ, B_μ being two sides of the triangle. For an indefinite metric with one time coordinate and the metric signature $(-+++)$ there are three types of bones: spacelike, null and timelike. Note that the form of the above relation has the opposite sign with respect to the relation for the area of a triangle in the three dimensional case, which follows from $2L = A \wedge B$. This is due to the fact that, if $B = (B_0, 0)$ and A is lying in the $z-$direction, then L lies in the $xy-$plane, thus its magnitude has to be thought of as a *real* quantity. Since in this case $A_\mu B^\mu = 0$, $B^2 < 0$ and $A^2 > 0$ it follows that $A^2 B^2 < 0$ but $L^2 > 0$ in order for L to have a real magnitude, [27]. Therefore the relation (1.6) is the appropriate one and we consider only timelike bones, such that the deficiencies are all real.

To get the field equations in this framework Regge considered approximating an Einstein space with the action

$$\mathcal{S} = \frac{1}{16\pi} \int R\sqrt{-g}\, d^4x$$

with a skeleton space using the variational principle for which $\mathcal{S}$ is calculated on the skeleton. Recalling the discussion at the beginning of this section, notice that there is no contribution to the action coming from the interior of the T_4 and T_3, this means that R is a distribution with support w. Hence, it can be expressed in terms of the deficiencies and areas of the bones, labelled by n:

$$\mathcal{S} = \frac{1}{8\pi} \sum_n L_n \epsilon_n.$$

Since the lengths l_p of all T_1^p contain the same information as the metric tensor, the variation of $\mathcal{S}$ is carried out varying these lengths. Regge proved in the appendix of [34] that it is possible to carry out the variation as if the deficiencies ϵ_n were constants, thus we obtain a set of field equations of the form[26]

$$\sum_n \epsilon_n \frac{\partial L_n}{\partial l_p} = 0$$

where p runs through all the 1-simplexes in the decomposition of the manifold. With the help of the law of cosines $l_p^2 = l_{p'}^2 + l_{p''}^2 - 2l_{p'}l_{p''}cos\theta_{pn}$, for the angle θ_{pn} opposite to l_p in T_2^n, the fact that $l_p = \sqrt{A_\mu A^\mu}$ for the edge A_μ, and $\frac{\partial L^2}{\partial l_p} = 2L\frac{\partial L}{\partial l_p}$ we obtain

$$\frac{\partial L_n}{\partial l_p} = \frac{1}{2} l_p ctg\theta_{pn}.$$

Hence, Einstein's equations[27] for an empty space in the Regge approximation are given by

$$\sum_n \epsilon_n ctg\theta_{pn} = 0, \tag{1.7}$$

where the sum runs over all 2-simplexes that have the p-th edge in common.

To summarize, Regge calculus is an approximation of a smoothly curved n-dimensional Riemannian manifold in terms of a collection of n-dimensional simplexes, each of them being flat in its interior and joined at their lower-dimensional faces. The curvature of the manifold is contained in the deficiency angles of the $(n-2)$-skeleton geometry together with its measure. By allowing the number of $(n-2)$-faces to go to infinity and keeping the limit (1.3) slowly varying one obtains the volume integral of the scalar curvature of the original smooth manifold.

1.3 Connection between General Relativity and Spin Networks

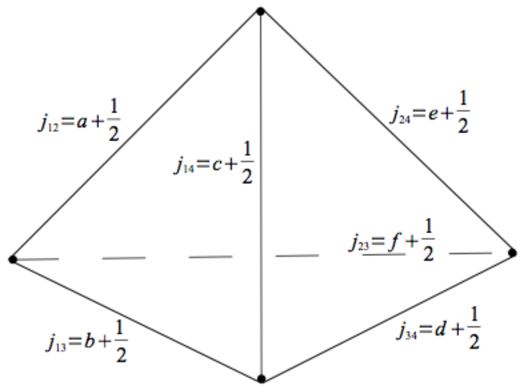
Once we have the concept of spin networks and the Regge calculus an immediate question regarding the relation between both arises. How can an object developed from basic notions of quantum theory for which there was no notion of space-time be related to a conceptually different theory, where space-time is the fundamental object? In 1969 G. Ponzano and T. Regge gave us some insight into this relation by explaining the relation between the asymptotic formula for the 6j-symbols, which are the coupling coefficients for a system consisting of three spins, and the path integral over the exponential of the integral over the Lagrangian density in a 3-dimensional Einstein theory, which resembles in a remarkable way a Feynman summation over histories as in QFT, [33]. In this section we will follow Ponzano's and Regge's

[26] Here we only consider the 4-dimensional case, but it can be generalized by taking L_n as the $(m-2)$-dimensional measure of T_{m-2}^n.

[27] It can be shown that these equations degenerate into $R_{\mu\nu} = 0$ for a differential manifold, [34]. Hence, the equations (1.7) describe the combinatorial version of an empty space without cosmological constant.

article leaving out the details and concentrating in the main points which are the asymptotic form and physical interpretation of the 6j-symbols, their generalization to 3nj-symbols and the rise of the "Feynman integral" out of the asymptotic formula.

Before we start describing the above mentioned relation, one should notice the association of the 6j-symbol $\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}$ to a tetrahedron T depicted by



where the length of an edge is chosen to be $a+\frac{1}{2}$ for a better approximation, rather than just a, to the length $[a(a+1)]^{\frac{1}{2}}$ for higher quantum numbers. A sufficient condition for the existence of a non-vanishing 6j-symbol is the restriction to those values of the angular momenta which satisfy the triangular inequalities for the set of three edges building a triangle, e.g. $|b-c| \leq a \leq b+c$. This is, however, not enough for the existence of T. With the help of Tartaglia's formula for the square of the volume V of the tetrahedron

$$2^3(3!)^2V^2 = \begin{vmatrix} 0 & j_{34}^2 & j_{24}^2 & j_{23}^2 & 1 \\ j_{34}^2 & 0 & j_{14}^2 & j_{13}^2 & 1 \\ j_{24}^2 & j_{14}^2 & 0 & j_{12}^2 & 1 \\ j_{23}^2 & j_{13}^2 & j_{12}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

we see that the condition $V^2 \geq 0$ is a necessary and sufficient condition for the existence of T. In what follows we will always assume that this condition is satisfied.[28]

1.3.1 Asymptotic formula for the 6j-symbols

The $6j$-symbols $\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{matrix} \right\}$ are defined as recoupling coefficients of the quantum mechanical coupling of three angular momenta[29] $\vec{j_1}$, $\vec{j_2}$ and $\vec{j_3}$ to $\vec{J}$. The defining relation is, similar to the Clebsch-Gordan coefficients, the change of basis from a given configuration of

[28] In [33] the case where $V^2 < 0$ is also treated in an analogous way to the WKB approximation method. In this region, the 6j-symbols have an exponential decrease, hence they do not vanish completely, but it can be interpreted as the impossibility of having six angular momenta forming a non-existing T.

[29] In the semiclassical limit we expect to be able to write the angular momenta as vectors in a three dimensional Euclidean space and the coupling as a vector sum of the angular momenta involved.

the angular momenta to another, both represent the two different ways of coupling $\overrightarrow{j_1}$, $\overrightarrow{j_2}$ and $\overrightarrow{j_3}$ to a given $\overrightarrow{J}$, by (1) first coupling $\overrightarrow{j_1}$ and $\overrightarrow{j_2}$ to $\overrightarrow{j_{12}}$ and *then* coupling this with $\overrightarrow{j_3}$ or (2) first coupling $\overrightarrow{j_2}$ and $\overrightarrow{j_3}$ to $\overrightarrow{j_{23}}$ and *then* coupling this to $\overrightarrow{j_1}$[30]. The resulting basis vectors for the angular momenta are in general different and are related by

$$|(j_1,\,(j_2,j_3)\,j_{23}),\,J\rangle=\sum_{j_{12}}\left[(2j_{12}+1)(2j_{23}+1)\right]^{\frac{1}{2}}(-1)^{j_1+j_2+j_3+J}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{array}\right\}|((j_1,j_2)\,j_{12},\,j_3),\,J\rangle\,.$$

Consider the option (1), then the probability that the sum of $\overrightarrow{j_2}$ and $\overrightarrow{j_3}$ has length in the interval $[j_{23},\, j_{23}+dj_{23}]$ is given by

$$(2j_{12}+1)(2j_{23}+1)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{array}\right\}^2 dj_{23} \tag{1.8}$$

On the other hand, if j_1, j_2, j_3, j_{12}, J are fixed the dihedral angle θ between the planes spanned by $\overrightarrow{j_1}$, $\overrightarrow{j_2}$ and $\overrightarrow{j_3}$, $\overrightarrow{J}$ is undetermined. Suppose that every configuration giving θ has equal probability $|d\theta|/(2\pi)$, then the probability for the length j_{23} to have the value between j_{23} and $j_{23}+dj_{23}$ is given by[31]

$$\left\{\frac{2}{(2\pi)}\left|\frac{d\theta}{dj_{23}}\right|\right\}dj_{23}=\frac{1}{6\pi}\frac{j_{12}j_{23}}{V} \tag{1.9}$$

where V is the volume of the tetrahedron spanned by all vectors involved and the following relation[32] was used

$$\frac{d\theta_{hk}}{dj_{rs}}=-\frac{j_{rs}j_{hk}}{6V}\ \text{for}\ h\neq k\neq r\neq s.$$

Comparing (1.8) with (1.9) we obtain for large angular momenta a result, due to Wigner, relating the volume V of a tetrahedron with the square of its corresponding $6j$-symbol:

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{array}\right\}^2\propto\frac{1}{24\pi V}$$

This result is, however, unacceptable since the numerical calculations show that the symbols are, in fact, rapidly oscillating functions of the indices. Hence, the above proportionality suggests that it is rather an approximation of the average of the 6j-symbol over a large enough interval of values its indices. Ponzano and Regge claimed that a correct approximation would be of the form

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{array}\right\}\simeq\frac{1}{\sqrt{12\pi V}}\mathfrak{C}$$

[30] The order of coupling determines the phase of the resulting state, cf. [14].

[31] The factor of 2 comes from the fact that there are two configurations giving the same value of j_{23}.

[32] This relation follows from

$$A_hA_k sin\theta_{hk}=\frac{3}{2}Vj_{hk}\ \text{and}\ cos\theta_{hk}=-\frac{9}{A_hA_k}\frac{\partial V^2}{\partial(j_{rs}^2)}\ \text{for}\ h\neq k\neq r\neq s$$

given in [33, Appendix B].

where $\mathfrak{C}$ is a rapidly oscillating function such that the average over a long interval of indices gives $\langle \mathfrak{C}^2 \rangle = \frac{1}{2}$.

In order to find out which form $\mathfrak{C}$ could have, Ponzano and Regge gave heuristic arguments, which we will give in a very short fashion. It is known, [14], that following relation between 6j-symbols and matrix representations[33] of the rotation group holds

$$\left\{ \begin{array}{ccc} c & a & b \\ f & b+\delta & a+\delta' \end{array} \right\} \simeq \frac{(-1)^{a+b+c+f+\delta}}{\sqrt{(2a+1)(2b+1)}} d^{(f)}_{\delta,\delta'}(\theta) \tag{1.10}$$

where a, b, $c \gg f$, δ, δ' and θ is the angle between the edges a and b intersecting at f and given by the law of cosines

$$cos\theta = \frac{a(a+1)+b(b+1)-c(c+1)}{2\sqrt{a(a+1)b(b+1)}}\ ;\ \text{for } 0 \leq \theta \leq \pi.$$

Relation (1.10) together with the Biedenharn-Elliott identity (2.19) give a result[34] suggesting the following general formula for the asymptotic form of the 6j-symbols

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} \simeq \frac{1}{\sqrt{12\pi V}} cos \left(\sum_{h,k=1}^{4} j_{hk}\theta_{hk} + \frac{\pi}{4} \right), \tag{1.11}$$

where $j_{hk} = j_{kh},\ j_{hh} = 0,$ and $\theta_{hk} = \theta_{kh}$ is given by

$$cos\theta_{hk} = -\frac{9}{A_h A_k} \frac{\partial V^2}{\partial (j^2_{rs})}$$

for $h \neq k \neq r \neq s$, and A_h is the area of the face opposite to the vertex h, cf. Footnote 32.

Remark. Even if (1.11) has no strict formal derivation, there are several arguments for accepting this relation:

1. Numerical accuracy which improves as the values of the angular momenta increase.
2. Invariance under the full symmetry group of the 6j-symbols.
3. The formula satisfies asymptotically the Biedenharn-Elliott identity as well as all identities which are enough to derive all properties and numerical values of the 6j-symbols.

1.3.2 The 3nj-symbols and their relation to general relativity

We consider now the generalizations of the $6j$-symbols for systems with a higher number of edges, called the $3nj$-symbols. These are best described by the use of diagrams that provide combinatorial information on how angular momenta are coupled. Since the coupling process is always the same[35] in a given scheme, where there are more than three angular

[33] Here the index f gives the dimension $(2f+1)$ of the representation and δ, δ' are the matrix indices.

[34] cf. Appendix C in [33].

[35] Two angular momenta are coupled, then a third is coupled to the resulting one, giving a new angular momenta which may be coupled to a forth one and so on; for instance, $(\overrightarrow{j_1} + \overrightarrow{j_2}) + \overrightarrow{j_3} = \overrightarrow{j_{12}} + \overrightarrow{j_3} = \overrightarrow{J}$.

momenta involved, the recoupling coefficients arising can be expressed in terms of 6j-symbols. Moreover, it is possible to write these symbols in terms of four 3j-symbols[36] as follows (cf. [14, Sec. 6.2])

$$\begin{aligned}\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\} &= \sum_{all\, m} \left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array}\right)\left(\begin{array}{ccc} j_1 & j_5 & j_6 \\ m_1' & m_5 & m_6' \end{array}\right) \times \\ &\times \left(\begin{array}{ccc} j_4 & j_2 & j_6 \\ m_4' & m_2' & m_6 \end{array}\right)\left(\begin{array}{ccc} j_4 & j_5 & j_3 \\ m_4 & m_5' & m_3' \end{array}\right)\left(\begin{array}{c} j_1 \\ m_1\, m_1' \end{array}\right)\left(\begin{array}{c} j_2 \\ m_2\, m_2' \end{array}\right) \times \\ &\times \left(\begin{array}{c} j_3 \\ m_3\, m_3' \end{array}\right)\left(\begin{array}{c} j_4 \\ m_4\, m_4' \end{array}\right)\left(\begin{array}{c} j_5 \\ m_5\, m_5' \end{array}\right)\left(\begin{array}{c} j_6 \\ m_6\, m_6' \end{array}\right),\end{aligned}$$

where

$$\left(\begin{array}{c} j \\ m\, m' \end{array}\right) = (-1)^{j+m}\delta_{m,-m'}.$$

Furthermore, it is always possible to evaluate a recoupling graph of a compact semi-simple group G as a sum of products of Racah coefficients of G. This result is known by the name of **Decomposition Theorem**, cf. [28, sec. 4.2] and section 4.2.1. Hence, a diagram D is essentially a clear notation for the expansion of a 3nj-symbol $[D]$ in terms of 3j-symbols:

$$[D] = \sum_{all\, m} \left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array}\right)\cdots\left(\begin{array}{ccc} j_p & j_q & j_r \\ m_p & m_q & m_r \end{array}\right)\left(\begin{array}{c} j_r \\ m_r\, m_r' \end{array}\right)\left(\begin{array}{ccc} j_r & j_s & j_t \\ m_r' & m_s & m_t \end{array}\right)\cdots$$

In fact, D is a 2-dimensional simplicial complex with a 1-to-1 correspondence between 1- and 2-simplexes in D and angular momenta j_i and 3j-symbols respectively in the r.h.s. of $[D]$. Furthermore, the boundary of a face (i.e. a 2-simplex) is the sum of the three edges in the corresponding 3j-symbol and we disregard the orientation of the simplexes. From the general form of $[D]$, observe that there are 2n faces and 3n edges in D.[37] Since there is a certain degree of arbitrariness regarding the position of the points, regard diagrams as equivalent whenever they yield the same symbol $[D]$. If only topological properties are considered, we use the word diagram to denote the equivalent class. On the other hand a configuration is an element of the equivalent class, i.e. a diagram with additional information about the angles and/or length of the edges, which removes the ambiguities.

Now we will give a rough description on how to calculate the 3nj-symbol $[D]$ for any diagram D. First introduce a diagram $\mathfrak{D}(D)$ defined as a 3-dimensional simplicial complex with $\partial\mathfrak{D}(D) = D$. The vertices, edges and faces of $\mathfrak{D}(D)$ are called external if they belong to the boundary, otherwise internal. We denote the cells of $\mathfrak{D}(D)$, which are 3-simplexes, by $T_k\,(k = 1, 2, \ldots, p)$ and all internal edges are labelled by $x_i\,(i = 1, 2, \ldots, q)$ and external edges by $l_j\,(j = 1, 2, \ldots, r)$. To each cell T_k we associate a function $[T_k]$ of the internal and,

[36] The 3j-symbols where defined by Wigner as the recoupling coefficients of three angular momenta to a zero one. In the diagrammatic notation, they correspond to a trivalent vertex as in [14], or its dual diagram, a triangle, which is easier to relate to the vector coupling picture of three vectors adding up to a zero vector.

[37] The value of n is related to the Euler characteristic:

$$\chi = f - e + v = n + v$$

for f faces, e edges and v vertices.

if the case is given, external edges, which constitute the diagram T_k. Then, form the product of all symbols $[T_k]$, the resulting function will be a function of all internal labels:

$$A(x_1, \ldots, x_q) := \prod_{k=1}^{p} [T_k] \, (-1)^{\chi} \prod_{i=1}^{q} (2x_i + 1)$$

where[38] $\chi = \sum_{j=1}^{q} (n_j - 2)x_j + \chi_0$. The constant χ_0 is a fixed phase to make χ an integer and n_j is the number of tetrahedra T with the common edge x_j.

Now, consider the sum over all internal variables

$$S := \sum_{x_1, \ldots, x_q} A(x_1, \ldots, x_q)$$

If there are no internal vertices, a case which is always possible to realize, then the sum is finite and $S = [D]$. On the other hand, in the more general case where there are internal vertices, the sum becomes infinite but in some simple cases it is possible to renormalize[39] it via a method described below to obtain $[D]$. This case is interesting and deserves our attention since it gives a result which suggests a formal analogy with the Feynman path integral formalism in connection with the theory of relativity via Regge calculus. Before discussing briefly this result, which is the most important one of this section, we conclude the description of the calculation of $[D]$ in the case where there are no internal vertices.

If we replace $[T_k]$ with the asymptotic formula (1.11) and express the cosine in terms of Euler functions, then $A(x_1, \ldots, x_q)$ will be the sum of 2^p pairwise conjugate terms. We express also the factor $(-1)^{\chi}$ in terms of the exponential function[40] $e^{\pm i\pi\chi}$ such that A takes following form when we denote by x_{ik} the edge x_i belonging to the tetrahedron T_k:

$$A(x_1, \ldots, x_q) = \frac{e^{i\pi\chi_0}}{2^p (12\pi)^{p/2} (V_1 \ldots V_p)^{1/2}} \left(\prod_{j=1}^{q} (2x_j + 1) e^{\pm i\pi(n_j - 2)x_j} \right) \times$$
$$\times \prod_{k=1}^{p} \left(e^{i[\sum x_{ik}\theta_{ik} + \pi/4]} + e^{-i[\sum x_{ik}\theta_{ik} + \pi/4]} \right)$$

which can be rearrange as a sum of terms[41] of the form proportional to

$$\prod_{j=1}^{q} (2x_j + 1) exp \left\{ i \left[\sum_{k=1}^{p_j} (\pm\theta_{kj} - \pi) + 2\pi \right] x_j \right\}.$$

[38]This formula for the Euler characteristic is only true if we assume that D and $\mathfrak{D}(D)$ are homeomorphic to S^2 and the 3-ball respectively, which is assumed in the following.

[39]In fact, the general case in which there are internal vertices was not discussed in [33], but only the simple case of a tetrahedron with an internal vertex was shown to converge with the renormalization method suggested by Regge and Ponzano and described in this section. A counter-example in which the limit of the renormalized partition function via this method does not converge is given in [6, Sec. 2.4]. For an extensive discussion of different methods of renormalization of the Regge-Ponzano partition function the reader is referred to [6].

[40]Even if this procedure is only correct for integer x_j, it is possible to extended it to half-integers as well.

[41]The summation in the exponential is over all tetrahedra T_k with the common internal edge x_j.

Then S will be the sum over the internal edges and over the 2^p terms similar to the above one. Hence we have

$$S = \sum_{2^p\, terms} \sum_{x_1} \cdots \sum_{x_q} \frac{C}{(V_1 \ldots V_p)^{1/2}} \prod_{j=1}^{q} (2x_j + 1) exp\left\{ iF(\theta_{kj}) x_j \right\},$$

where $F(\theta_{kj})$ is a linear function of all dihedral angles of the tetrahedra in $\mathfrak{D}(D)$. If we replace the summation over all internal variables by integrals, one should expect that the most important contributions arise from points where the phase is stationary with respect to the x_j's, i.e. $F(\theta_{kj}) \stackrel{!}{=} 0$, thus we have

$$\sum_{k=1}^{p_j} (\pi \mp \theta_{kj}) \stackrel{!}{=} 2\pi,$$

meaning that the sum of internal angles around x_j is 2π, which implies the existence of a configuration embedded in a three-dimensional Euclidean space. Since each solution[42] of these stationary conditions represent a specific configuration. we obtain as final result the sum of contributions from each configuration. Notice that we assumed already at the beginning of the discussion for the calculation of S that the diagram D was embeddable in the 2-sphere. We did this by fixing the form of the Euler characteristic, cf. footnote 38, thus it is not surprising that the solutions of the stationary conditions give configurations embeddable in a 3-dim Euclidean space; χ contains information about the combinatorial manifold. The main idea here is that through solutions of these stationary conditions for situations with general Euler characteristics we obtain configurations of D corresponding, in the limit, to differentiable manifolds.

Finally, consider the sum S in a simple case where there are internal vertices. In this case the sum is infinite, [33, 28]. However the form of the factor responsible for this infinity makes it easy to renormalize the sum such that for general $[D]$ we have

$$[D] = \lim_{R \to \infty} \frac{1}{\mathfrak{R}(R)^P} \sum_{x_1 < R} \cdots \sum_{x_q < R} A(x_1, \ldots, x_q)$$

where P is the number of internal vertices and $\mathfrak{R}(R) \simeq \frac{1}{\pi}\left(\frac{4\pi R^3}{3}\right)$ is the factor in S responsible for the infinity when $R \to \infty$.

Furthermore, assume that the number of vertices and edges in D and $\mathfrak{D}(D)$ is very high such that the simplicial complex approaches a differential manifold M with boundary D. According to Regge the sum $\sum_{j=1}^{q}\left[\sum_{k=1}^{p_j}(\pi \mp \theta_{kj})\right] x_j$ which, as stated previously, gives a possible configuration of the embedding of D in some manifold, approaches $\mathcal{S}(M) = \frac{1}{16\pi}\int_M RdV$ where R is the scalar curvature of M. Therefore the positive frequency part of S has the following form

$$S^+ = \frac{1}{\mathfrak{R}^P} \int_{D\, fixed} e^{i\mathcal{S}(M)} d\mu(M), \tag{1.12}$$

[42] Since we assumed that there are no internal vertices, the internal edges in $\mathfrak{D}(D)$ connect external vertices of D and they are sufficient to specify completely the configuration.

where the summations over the internal variables x_j were interpreted as an integration over all manifolds with a given fixed boundary $\partial M = D$. The above integral is not defined in any precise mathematical sense since its derivation was heuristic in nature and its merely function is to show the strong resemblance with a Feynman path integral with the same Lagrangian density $\mathcal{L}$ as in a 3-dimensional Einstein theory.

The discussion above is the link between the concepts exposed in the previous sections and it gives us an idea of the possible, not yet fully understood, relation between the abstract idea of space and basic quantum mechanical objects such as quantum angular momenta. This suggests the possibility of describing the notion of space in a physical manner, where a richer structure than the one given by the theory of general relativity arises. In Chapter 3 we will discuss the more general concepts formalizing the discussion above about decompositions of 3-manifolds and the invariants defined out of their combinatorial properties, for instance, in the above case the Feynman path integral over all differential 3-manifolds with a given boundary.

Chapter 2

Mathematical Framework

The mathematical framework encoded in the diagrammatic language of spin networks is related to the category of representations of the (deformed) quantum enveloping Hopf algebra of the sl_2 Lie algebra, denoted $U_q(sl_2)$. In this section the mathematical concepts are introduced briefly.

First, the concept of Hopf algebras (which are bialgebras with an antipode obeying some identities) is described as in [25]. This structure is a generalization of the concept of a group with a linear map, the antipode, sending g to its inverse g^{-1}. The quantum enveloping algebra $U(\mathfrak{g})$ is then a non-commutative Hopf algebra generated by 1 and the generators of the Lie algebra $\mathfrak{g}$. This algebra can then be "deformed" by a non-zero parameter q which enters the commutation relations of the generators of $U(\mathfrak{g})$ in such manner that if $q \to 1$, the deformed enveloping algebra reduces to the bialgebra $U(\mathfrak{g})$.

Second, the main ideas of category theory are introduced in order to understand the relation between spin networks and the category of representations of $U_q(sl_2)$, which is a monoidal category $\mathcal{C}$ with some extra structure. The monoidal structure of the category is given by the tensor product of representations and the morphisms (which are intertwiners between representations) can be expressed in a graphical way. The extra structure is given by the dual-functor $\star : \mathcal{C}^{op} \to \mathcal{C}$, where $\mathcal{C}^{op}$ is the canonical dual of the category $\mathcal{C}$. This structure makes the category into a monoidal category with duals and allows it to have a pivotal structure, which is a morphism $\epsilon_A : e \to A \otimes A^\star$ with some axioms and compatibility with the counit and the tensor product. The pivotal property is equivalent to the requirement of planar isotopy of the diagrams in the framework of spin networks, cf. [7, 8] and [26]. If a pivotal category is invariant under diffeomorphisms of S^2 it is called spherical. Ultimately, it is this type of categories which give the general framework for spin networks and the category of representations of $U_q(sl_2)$ is a special case which give rise to the Turaev-Viro invariant described in Chapter 3.

Third, the diagrammatic language is given by the Temperley-Lieb recoupling theory which is a powerful tool to evaluate networks and gives the identification of the tetrahedron with the recoupling coefficients as well as basic identities between (quantum) $6j$-symbols such as the Biedenharn-Elliott and orthogonality relations. In this context, we also give the formal definition of many of the concepts already seen in Section 1.1, for instance, the projectors or n-edges and its loop-value.

To finalize this chapter, we give a very brief and general account of the topological quan-

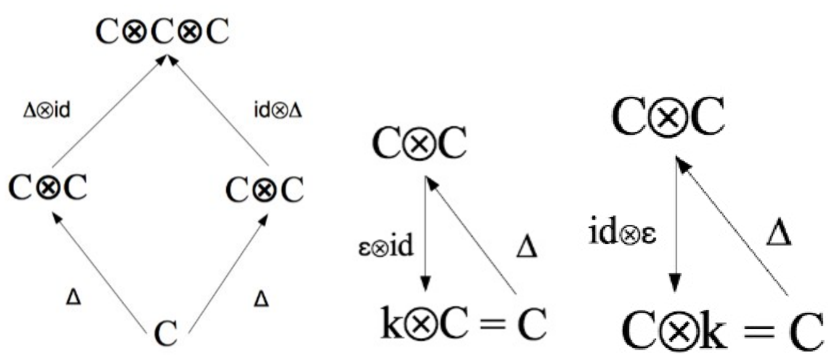
tum field theory which relates, in our context, concepts of combinatorial manifolds with the algebraical data needed to construct the invariants of 3-manifolds. We will encounter this theory explicitly in Chapter 3 where the state sum invariant of $U_q(sl_2)$ is described.

2.1 Hopf algebras and Quantum Groups

2.1.1 Algebras, Coalgebras and Bialgebras

An algebra $A(\cdot, +; k)$ over a field k is a ring $(A, \cdot, +)$ which is also a vector space $(A, +; k)$ and the action of the field k on A is compatible with the product and addition. The associativity of the product and the existence of a unit element (if required) are expressed as commutative diagrams[1]. In this language, an algebra is a vector space A with a product (and a unit) such that these diagrams commute. Here we will always assume the existence of a unit. If A, B are algebras, then $A \otimes B$ is an algebra which has a vector space given by the tensor product of the vector spaces A, B and a product defined by $(a \otimes b)(c \otimes d) \equiv (ac \otimes bd)$.

The dual notion of an algebra is a coalgebra $(C, +, \Delta, \epsilon; k)$ over a field k, which is a vector space $(C, +; k)$ and a linear map $\triangle$: $C \to C \otimes C$, called the coproduct, which is coassociative and for which there exist a linear map $\epsilon : C \to k$, called the counit. The commutative diagrams for coassociativity (on the left) and counit (the two on the right) are the following



Hence, since these diagrams commute, coassociativity means $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ and the existence of a counit ϵ means $(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$. If we denote, for any $c \in C$,

$$\Delta(c) = \sum_i c_{i(1)} \otimes c_{i(2)} \in C \otimes C$$

the coassociativity takes the explicit form

$$\sum_i c_{i(1)} \otimes c_{i(2)(1)} \otimes c_{i(2)(2)} = \sum_i c_{i(1)(1)} \otimes c_{i(1)(2)} \otimes c_{i(2)}$$

and the counit means explicitly

$$\sum_i \epsilon(c_{i(1)}) \cdot c_{i(2)} = c$$

[1]For the definition cf. Sec.2.2.

by the isomorphism $k \otimes C \cong C$. From now on we will drop the sum symbol and index and think of the sum implicitly in the notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$.

If a vector space $(H, +; k)$ is both an algebra and a coalgebra and both structures are compatible, i.e. for all g, $h \in H$ we have

$$\Delta(hg) = \Delta(h)\Delta(g),\ \Delta(1_H) = 1_H \otimes 1_H,\ \epsilon(hg) = \epsilon(h)\epsilon(g),\ \epsilon(1_H) = 1_k,$$

then H is called a **bialgebra**. The compatibility of the algebra and coalgebra structures is due to the fact that Δ, ϵ are algebra maps and $\cdot : H \otimes H \to H$, and $\eta : k \to H$ are coalgebra maps, i.e. respect the (co-)algebra structure. The map $\cdot : H \otimes H \to H$ is the algebra multiplication and the map $\eta : k \to H$ is a map used to express diagrammatically the existence of a unit element in H (cf. Sec. 2.2).

2.1.2 Hopf algebras and some properties

Definition 9. A **Hopf algebra** H is a bialgebra with a linear map $S : H \to H$ called **antipode**, which obeys $\cdot(S \otimes id) \circ \Delta = \cdot(id \otimes S) \circ \Delta = \eta \circ \epsilon$. The axiom for the antipode is expressed as a commutative diagram in the following way

$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & k \arrow[r, "\eta"] & H \\
H\otimes H \arrow[rr, "id\otimes S", "S\otimes id"'] & & H\otimes H \arrow[u]
\end{tikzcd}$$

Remark. Although the antipode is the generalization of the concept of inverse in a group, it is not required that $S^2 = id$ or the existence of S^{-1}, however, in the finite dimensional case S^{-1} always exists. Explicitly, we have from the counit and antipode axioms the relations

$$h = h_{(1)}\epsilon(h_{(2)}) = \epsilon(h_{(1)})h_{(2)}$$

and

$$h_{(1)}S(h_{(2)}) = \epsilon(h) \cdot 1_H = S(h_{(1)})h_{(2)};\ S(h) = \epsilon(h_{(1)})S(h_{(2)}),$$

for any $h \in H$.

The uniqueness of the antipode S of a given Hopf algebra follows from the counit and antipode axioms and from the linearity of S. The antipode is an anti-algebra and anti-coalgebra map, i.e. it transposes the order of the factors; this means that, for any g, $h \in H$, S obeys $S(gh) = S(h)S(g)$, $S(1) = 1$ and $(S \otimes S) \circ \Delta h = \tau \circ \Delta \circ Sh$, $\epsilon Sh = \epsilon h$.

Example 10. The **Weyl algebra**. Consider the generators 1, X, g, g^{-1} of an algebra with relations $gg^{-1} = 1 = g^{-1}g$, $g^{-1}Xg = qX$, where $q \in k$ is fixed and invertible. These become a Hopf algebra with following relations

$$\Delta X = X \otimes 1 + g \otimes X;\ \Delta g = g \otimes g,\ \Delta g^{-1} = g^{-1} \otimes g^{-1}$$

$$\epsilon(X) = 0,\ S(X) = -g^{-1}X;\ \epsilon(g) = 1 = \epsilon(g^{-1}),\ S(g) = g^{-1},\ S(g^{-1}) = g$$

Remark. i) The defining operations are S and Δ, since with $\epsilon(h) = S(h_{(1)})h_{(2)}$, the counit follows directly from those operators, e.g. $\epsilon(X) = S(X)1 + S(g)X = -g^{-1}X + g^{-1}X = 0$. *ii)* The comultiplication Δ and the antipode extend as algebra and anti-algebra maps respectively, since $\Delta Xg = Xg \otimes g + g^2 \otimes Xg = \Delta X \Delta g$ and they respect the defining relations, e.g. $\Delta Xg = q\Delta gX$.

Example 11. The **universal enveloping Hopf algebra** of $\mathfrak{g}$, denoted $U(\mathfrak{g})$:

Let $\mathfrak{g}$ be a Lie algebra over a field k, then the non-commutative algebra $U(\mathfrak{g})$ generated by 1 and the generators of $\mathfrak{g}$ is a non-commutative Hopf algebra with coproduct, counit and antipode given by $\Delta\xi = \xi \otimes 1 + 1 \otimes \xi$, $\epsilon(\xi) = 0$, $S(\xi) = -\xi$ for any $\xi \in \mathfrak{g}$ extended as (anti-)algebra maps consistent with $\Delta[\xi, \eta] = [\xi, \eta] \otimes 1 + 1 \otimes [\xi, \eta]$, $\epsilon([\xi, \eta]) = 0$ and $S([\xi, \eta]) = -[\xi, \eta]$.

Remark. Since the notion of inverse in group theory is generalized as antipode in the theory of Hopf algebras, the category of finite groups extends to that of Hopf algebras. Similarly there is an association from the category of Lie algebras to that of universal enveloping Hopf algebras given by a forgetful functor, cf. Section 2.2.

The action of a Hopf algebra H on a vector space V is defined in the usual way. An algebra A is an H-module algebra if A is a left H-module and the action of H for any $h \in H$, a, $b \in A$ is as follows

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b),\ h \triangleright 1 = \epsilon(h) \cdot 1$$

A coalgebra C is a left H-module coalgebra if C is an H-module and for any $h \in H$, $c \in C$

$$\Delta(h \triangleright c) = \sum h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleright c_{(2)},\ \epsilon(h \triangleright c) = \epsilon(h)\epsilon(c)$$

This means that the action $\triangleright$: $H \otimes C \to C$ is a coalgebra map, i.e. $\Delta(h \triangleright c) = (\Delta h) \triangleright (\Delta c)$.

Example 12. The adjoint action for $H = U(\mathfrak{g})$ is for all $\xi, \eta \in \mathfrak{g}$, $\xi \triangleright \eta = [\xi, \eta]$. The action of the universal enveloping Hopf algebra of $\mathfrak{g}$ on a $U(\mathfrak{g})$-module algebra A is the usual notion (as for Lie algebras), $\xi \triangleright (ab) = (\xi \triangleright a)b + a(\xi \triangleright b)$, $\xi \triangleright 1 = 0$ for any $\xi \in \mathfrak{g}$.

If a Hopf algebra H acts on vector spaces V and W, then it also acts on the tensor product space $V \otimes W$ by

$$h \triangleright (v \otimes w) = \sum h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w,\ \ \forall h \in H,\, v \in V,\, w \in W$$

Hence, the two actions have a tensor product and the tensor product of H-modules is also an H-module. If a map $\phi : V \to W$ commutes with the corresponding actions, i.e. $\phi(h \triangleright v) = h \triangleright \phi(v)$ for all $v \in V$, then ϕ is called an intertwiner. This type of maps constitute the morphisms in the category of Hopf algebra representations, which are closely related to the theory of spin networks.

The dual notion of commutativity is cocommutativity, i.e. $\tau \circ \Delta = \Delta$, where $\tau : H \otimes H \to H \otimes H$ is the transposition map and can be weakened with the help of an element $\mathcal{R} \in H \otimes H$ called the quasitriangular structure. Then, the Hopf algebra H is cocommutative only up to conjugation by $\mathcal{R}$.

Definition 13. A **quasitriangular bialgebra** is a pair $(H, \mathcal{R})$ where H is a bialgebra and $\mathcal{R} \in H \otimes H$ is invertible and obeys

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\,,\ (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.1}$$

$$\tau \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1},\ \forall h \in H \tag{2.2}$$

With the notation $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, $\mathcal{R}_{ij} \in H \otimes H \otimes ... \otimes H$ is defined as

$$\mathcal{R}_{ij} = \sum 1 \otimes \ldots \otimes \underset{i-th}{\mathcal{R}^{(1)}} \otimes \ldots \otimes \underset{j-th}{\mathcal{R}^{(2)}} \otimes \ldots \otimes 1$$

To understand Eq. (2.1) let us write the first equation explicitly. On the l.h.s. we have

$$(\Delta \otimes id)\mathcal{R} = \sum_i (\Delta \mathcal{R}_i^{(1)}) \otimes \mathcal{R}_i^{(2)} = \sum_i (\sum_j \mathcal{R}^{(1)}_{(1)i,j} \otimes \mathcal{R}^{(1)}_{(2)i,j}) \otimes \mathcal{R}_i^{(2)}$$

and with $\mathcal{R}_{13} = \sum_m \mathcal{R}_m^{(1)} \otimes 1 \otimes \mathcal{R}_m^{(2)}$ and $\mathcal{R}_{23} = 1 \otimes \mathcal{R} = \sum_k 1 \otimes \tilde{\mathcal{R}}_k^{(1)} \otimes \tilde{\mathcal{R}}_k^{(2)}$ we have on the r.h.s.

$$\mathcal{R}_{13}\mathcal{R}_{23} = (\sum_m \mathcal{R}_m^{(1)} \otimes 1 \otimes \mathcal{R}_m^{(2)})(\sum_k 1 \otimes \tilde{\mathcal{R}}_k^{(1)} \otimes \tilde{\mathcal{R}}_k^{(2)}) = \sum_m \sum_k \mathcal{R}_m^{(1)} \otimes \tilde{\mathcal{R}}_k^{(1)} \otimes \mathcal{R}_m^{(2)}\tilde{\mathcal{R}}_k^{(2)}$$

Comparing the r.h.s. and the l.h.s. we get

$$\mathcal{R}^{(1)}_{(1)i,j} = \mathcal{R}_m^{(1)},\ \mathcal{R}^{(1)}_{(2)i,j} = \tilde{\mathcal{R}}_k^{(1)},\ \mathcal{R}_i^{(2)} = \mathcal{R}_m^{(2)}\tilde{\mathcal{R}}_k^{(2)}\ \ for\, some\, i,\ j,\ m,\ k.$$

This means that the factors of the coproduct of $\mathcal{R}$ are the factors of itself -in the first and second position- or a product of its factors, in the third position.

The quasitriangular structure $\mathcal{R}$ in a bialgebra H obeys $\cdot(\epsilon \otimes id)\mathcal{R} = \cdot(id \otimes \epsilon)\mathcal{R} = 1_H$. If H is a Hopf algebra then, it also obeys $(S \otimes id)\mathcal{R} = \mathcal{R}^{-1}$ and $(id \otimes S)\mathcal{R}^{-1} = \mathcal{R}$. Hence it also satisfies $(S \otimes S)\mathcal{R} = \mathcal{R}$.

The defining characteristics of a quasitriangular bialgebra imply the abstract quantum Yang-Baxter-Equation,

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.3}$$

Also, if $(H,\, \mathcal{R})$ is a quasitriangular Hopf algebra, the antipode S is automatically invertible and satisfies $S^2(h) = uhu^{-1}$, $\forall h \in H$, where u is an invertible element in H obeying

$$u = \sum (S\mathcal{R}^{(2)})\mathcal{R}^{(1)},\ u^{-1} = \sum \mathcal{R}^{(2)} S^2 \mathcal{R}^{(1)},\ \Delta u = Q^{-1}(u \otimes u)$$

with $Q = \mathcal{R}_{21}\mathcal{R}$. Similarly, the antipode of the previous element $v = Su$ satisfies $S^{-2}(h) = vhv^{-1}$, $\forall h \in H$ and obeys

$$v = \sum \mathcal{R}^{(1)} S\mathcal{R}^{(2)},\ v^{-1} = \sum (S^2\mathcal{R}^{(1)})\mathcal{R}^{(2)},\ \Delta v = Q^{-1}(v \otimes v).$$

Having $u,\, v\, \in\, H$, the element $uv = vu$ is central, i.e. commutes with all elements[2] of H, and obeys the relation $\Delta(uv) = Q^{-2}(uv \otimes uv)$. On the other hand, the element $w = uv^{-1} = v^{-1}u$ is group-like and implements S^4 by conjugation[3]. If the element uv has a central square root called the ribbon element ν, i.e. $\nu^2 = uv$, satisfying $\Delta\nu = Q^{-1}(\nu \otimes \nu)$, $\epsilon(\nu) = 1$, $S(\nu) = \nu$, then the quasitriangular Hopf algebra is called a **ribbon Hopf algebra** (cf. Sec. 2.2.3).

[2]This follows from the fact that $\forall h \in H:\ S^2(h) = uhu^{-1}$ and $S^{-2}(h) = vhv^{-1}$, thus, $h = (vu)h(vu)^{-1}$.
[3]From $h = v^{-1}S^{-2}(h)v$, we have $S^2(S^{-2}(h)) = v^{-1}S^{-2}(h)v$, thus, v^{-1} implements S^2 as well.

Example 14. The **anyon-generating quantum group** is generated by 1, g, g^{-1} and the relation $g^n = 1$ with the Hopf algebra structure $\Delta g = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$. It has the following non-trivial quasitriangular structure

$$\mathcal{R} = \frac{1}{n} \sum_{a,b=0}^{n-1} e^{-\frac{2\pi i ab}{n}} g^a \otimes g^b$$

The r.h.s. of the first equation in (2.1) is

$$\mathcal{R}_{13}\mathcal{R}_{23} = \frac{1}{n^2} \sum_{a,b,c,d} e^{-\frac{2\pi i(ab+cd)}{n}} g^a \otimes g^c \otimes g^{b+d}$$

which can be simplify with $b\acute{} = b + d$ and $n^{-1} \sum_{b=0}^{n-1} e^{-\frac{2\pi i ab}{n}} = \delta_{a,0}$.

Since the notion of an algebra is dual to the one of a coalgebra, the axioms of a Hopf algebra are self dual when interchanging Δ, ϵ and $\cdot$, η. In terms of dual linear spaces, this symmetry gives for every finite dimensional Hopf algebra H a dual Hopf algebra H^* build on the vector space dual to H (cf. Sec. 1.4 in [25]).

The axioms of a quasitriangular Hopf algebra are *not self-dual* but it is possible to extend the concept of a dual quasitriangular structure as an "invertible" map $A \otimes A \to k$, where A is the Hopf algebra in the dual formulation of H; the invertibility being defined in a suitable way (cf. Sec. 2.2 in [25]).

The above description of Hopf algebras and quasitriangular structure is intended to present briefly the quantum group $U_q(sl_2)$ and it is not the intention to give a detailed description of any of the concepts of this broad topic since this would go beyond the scope of this dissertation.

2.1.3 The Quantum Group $U_q(\mathfrak{sl}_2)$

Recall that the smallest simple Lie algebra is $\mathfrak{sl}_2$ with the following relations for the operators H and X_+, X_-

$$[H,\, X_\pm] = \pm 2X_\pm,\ [X_+,\, X_-] = H$$

The deformation of the universal enveloping algebra of $\mathfrak{sl}_2$ (cf. Example 11) with a parameter $q \neq 0$ defines the quantum group $U_q(\mathfrak{sl}_2)$, which is a non-commutative Hopf algebra generated by 1, X_+, X_-, $q^{H/2}$, $q^{-H/2}$ and the relations

$$q^{\pm H/2} q^{\mp H/2} = 1,\ q^{H/2} X_\pm q^{-H/2} = q^{\pm 1} X_\pm,\ [X_+,\, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{2.4}$$

Similar to example 10, this forms a Hopf algebra with comultiplication, counit and antipode maps as follows

$$\Delta q^{\pm H/2} = q^{\pm H/2} \otimes q^{\pm H/2},\ \Delta X_\pm = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm$$

$$\epsilon q^{\pm H/2} = 1,\ \epsilon X_\pm = 0;\ SX_\pm = -q^{\pm 1} X_\pm,\ Sq^{\pm H/2} = q^{\mp H/2}$$

The relations between the generators are defined such that if $q \to 1$ (2.4) reduce to the ones defining the Lie algebra $\mathfrak{sl}_2$. For example using the l'Hôpital rule for the third equation of (2.4) we have

$$[X_+,\, X_-] = \lim_{q\to 1} \frac{q^H - q^{-H}}{q - q^{-1}} = \lim_{q\to 1} \frac{H(q^{H-1} + q^{-H-1})}{1 + q^{-2}} = H.$$

The parameter q is usually an element of $\mathbb{C}$, however, if no calculations are needed it is useful to work over the field of formal power series of a parameter t, denoted $\mathbb{C}\,[t]$ and we define $q = e^{t/2}$. If some calculations are needed it is not possible to work over $\mathbb{C}\,[t]$ since q may not have a finite value (the formal power series might not converge) and the precise value of q is relevant for some aspects of the theory, e.g. the case when q is a root of unity, cf. Section 2.3 and Chapter 3.

The above defined Hopf algebra has a quasitriangular structure

$$\mathcal{R} = q^{\frac{H\otimes H}{2}} \exp_{q^{-2}} \left\{(1 - q^{-2})q^{H/2} X_+ \otimes q^{-H/2} X_-\right\}$$

where the q-exponential is defined as

$$e^x_{q^{-2}} = \sum_{n=0}^{\infty} \frac{x^n}{[n;\, q^{-2}]!}\ ;\ \left[n;\, q^{-2}\right] = \frac{1 - q^{-2n}}{1 - q^{-2}}\ ,\ \left[n;\, q^{-2}\right]! = \left[n;\, q^{-2}\right] \ldots \left[1;\, q^{-2}\right] \tag{2.5}$$

and follows similar properties as the usual exponential function with operators (cf. [25], p. 86).

The quantum group $U_q(\mathfrak{sl}_2)$ acts on objects similar to the Lie algebra $\mathfrak{sl}_2$-modules. In fact, for $q\in\mathbb{R}$ and each $j = 0,\, \frac{1}{2},\, 1, \ldots$ the real form $U_q(\mathfrak{su}_2)$ has a $(2j + 1)$-dimensional unitary irreducible representation $V_j = \{|j, m > \,|\, m = -j, \ldots, j\,\}$such that

$$X_\pm|j, m >= \sqrt{[j \mp m]\,[j \pm m + 1]}|j, m \pm 1 >;\ q^{H/2}|j, m >= q^m|j, m > \tag{2.6}$$

If q is a **primitive root of unity**[4] these formulas also apply, however, the number j has to be replaced by a "quantum integer" $[j]$ so the allowed range of j is restricted in a suitable way, cf. Chapter 3. For now, we will work over $\mathbb{C}$ and denote the generators by K, K^{-1}, $X_\pm$ instead of $q^{H/2}$, $q^{-H/2}$, $X_\pm$. Then the quotient of $U_q(\mathfrak{sl}_2)$ denoted by $U_q^{(r)}(\mathfrak{sl}_2)$ is the finite dimensional quasitriangular Hopf algebra generated by 1, K, K^{-1}, $X_\pm$ with relations

$$K^{\pm1}K^{\mp1} = 1,\ KX_\pm K^{-1} = q^{\pm1}X_\pm,\ [X_+,\, X_-] = \frac{K^2 - K^{-2}}{q - q^{-1}}\ ;\quad \underbrace{K^{4r} = 1,\ X_\pm^r = 0}_{relations\, which\, quotient\, U_q(\mathfrak{sl}_2)}$$

It has the operations inherited from $U_q(\mathfrak{sl}_2)$ but a different quasitriangular structure given by

$$\mathcal{R} = \mathcal{R}_K \sum_{m=0}^{r-1} (KX_+)^m \otimes (K^{-1}X_-)^m \frac{(1 - q^{-2})^m}{[m;\, q^{-2}]!}\ ;\ \mathcal{R}_K = \frac{1}{4r} \sum_{a,b=0}^{4r-1} q^{-ab/2} K^a \otimes K^b.$$

[4]Here, primitive means that $n > 1$ and there is no smaller n such that $q^n = 1$.

The representations V_j of $U_q^{(r)}(\mathfrak{sl}_2)$ are the same as in (2.6), however only for spins in the range $j = 0, \frac{1}{2}, 1, \ldots, \frac{r-1}{2}$. To see that, notice that if q is a primitive root of unity with $q^r = 1$ the quantum integer is $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \neq 0$ for $n = 1, \ldots, r-1$ but $[r] = 0$, so the action of $X_\pm$ is well-defined if j is in the allowed range $0 \leq j - m \leq r - 1$; $m = -j, \ldots, j$, cf. Section 2.3.1.

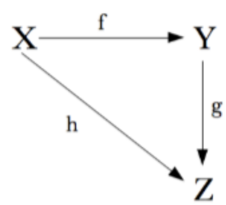
The above case is important since it allows the regularization of the Ponzano-Regge partition function, which otherwise diverges for some manifolds. The quantum group $U_q(sl_2)$ at a primitive root of unity causes the state sum to become finite, thus, it is possible to define an invariant for general 3-manifolds, as described in Chapter 3.

2.2 Category Theory

In this section, the main concepts of category theory will be introduced briefly following [24] closely. This theory is the general framework needed to understand the mathematics behind spin networks and its relation with the representations of the quantum group $U_q(\mathfrak{sl}_2)$. Category theory, roughly speaking, studies in an abstract way the properties of specific mathematical concepts by gathering them in collections of objects and arrows (or morphisms) which satisfy certain fundamental conditions. In other words, it is the theory for dealing, in the most general and abstract way, with concepts like sets, topological spaces, vector spaces, groups, etc.

2.2.1 Basic concepts of category theory

Let us start by introducing one of the basic concepts of category theory, the **commutative diagram**. Consider a diagram



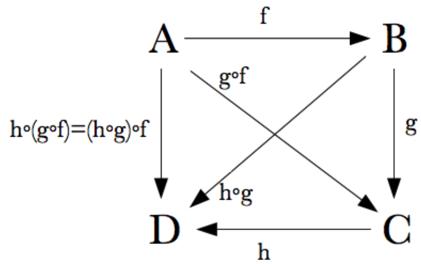
where $f : x \mapsto fx$, $h : x \mapsto hx$ and $g : y \mapsto gy$ are morphisms. If $h = g \circ f$ the diagram is called commutative.

Definition 15. A **category** $\mathcal{C}$ is a collection of objects $a, b, c, \ldots$ in $\mathcal{C}$ together with a collection $Hom_{\mathcal{C}}(a, b)$ of morphisms for any pair of objects a, b in $\mathcal{C}$ with a composition rule, such that for each a, b, c in $\mathcal{C}$ and each pair of arrows

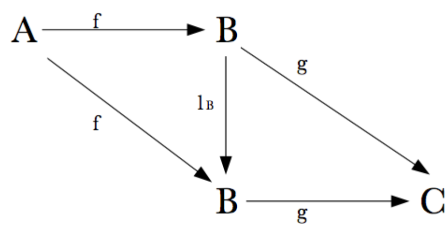
$$a \xrightarrow{f} b \text{ and } b \xrightarrow{g} c$$

there is a composite arrow $a \xrightarrow{g \circ f} c$, and for each object a in $\mathcal{C}$ there is an identity arrow $1_a : a \to a \in Hom_{\mathcal{C}}(a, a)$. The composition rule and the identity arrow are subject to the following axioms:

- Associativity: given the arrows f, g as above and $c \xrightarrow{h} d$, the composition is always associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$, so the following diagram is commutative



- Unit law: for all arrows $a \xrightarrow{f} b \xrightarrow{g} c$ the composition with the unit arrow 1_b yields $1_b \circ f = f$, $g \circ 1_b = g$, i.e. the following diagram commutes



The collections of morphisms are required to be pairwise disjunct, i.e. $Hom_{\mathcal{C}}(a, b) \cap Hom_{\mathcal{C}}(c, d) = \emptyset$ for any objects a, b, c, d with $a \neq c$ and $b \neq d$.

Example. Typical examples of categories are:

- **Set**, with objects all sets and morphisms all functions between them,
- **Grp**, where the objects are groups and the arrows are group homomorphisms,
- **Top**, the category of all topological spaces and continuous maps,
- **Vect(k)**, with vector spaces over a field k and linear maps,
- Let A be a unital algebra and ${}_A\mathcal{M}$ the **category of A-modules** with vector spaces, on which A acts, as objects and the morphisms are linear maps that commute with the action of A, i.e. intertwines.

Definition 16. A **functor** is a morphism between categories; more precisely, for categories $\mathcal{C}$, $\mathcal{B}$ a functor $T : \mathcal{C} \to \mathcal{B}$ consists of two related functions

1. **Object function** T: for any object c in $\mathcal{C}$, we have $c \mapsto Tc$, where Tc is in $\mathcal{B}$.
2. **Arrow function** T: for any morphism $f : c \to c'$ of $\mathcal{C}$, we have $f \mapsto Tf$, where $Tf : Tc \to Tc'$ is an arrow of $\mathcal{B}$.

such that $T(1_c) = 1_{Tc}$ and $T(g \circ f) = Tg \circ Tf$.

A **covariant** functor $T : \mathcal{C} \to \mathcal{B}$ respects the structure of the categories, i.e. for any two morphisms that can be composed T sends $f \mapsto Tf$ and $g \mapsto Tg$ such that $T(g \circ f) = Tg \circ Tf$. On the other hand, a **contravariant** functor $F : \mathcal{C} \to \mathcal{B}$ is defined as sending $f : a \to b$ to $Ff : Fb \to Fa$ such that $F(g \circ f) = Ff \circ Fg$ for any two morphisms f, g of $\mathcal{C}$ that can be composed, [25].

An example of a functor is the forgetful functor, which, as the name suggests, "forgets" some or all of the structure of an algebraic object in a given category, e.g. the functor $U : Grp \to Set$ assigns to each group G the set UG of its elements and each homomorphism the same function regarded as a function between sets.

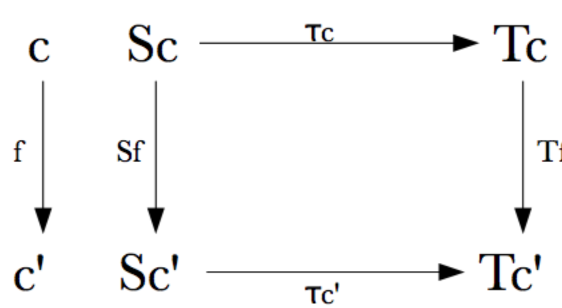
Given some functors $\mathcal{C} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{A}$, the composite functions on objects and arrows of $\mathcal{C}$ define a functor, i.e. functors may be composed. Furthermore, the composition is associative and for each category $\mathcal{C}$ there exists an identity functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which acts as an identity for the composition of functors. Hence, we can regard the collection of categories as a category itself where the functors are the morphisms of this category.

Definition. A functor $T : \mathcal{C} \to \mathcal{B}$ is an **isomorphism** if and only if there is a functor $S : \mathcal{B} \to \mathcal{C}$ for which $S \circ T = I_{\mathcal{C}}$ and $T \circ S = I_{\mathcal{B}}$.

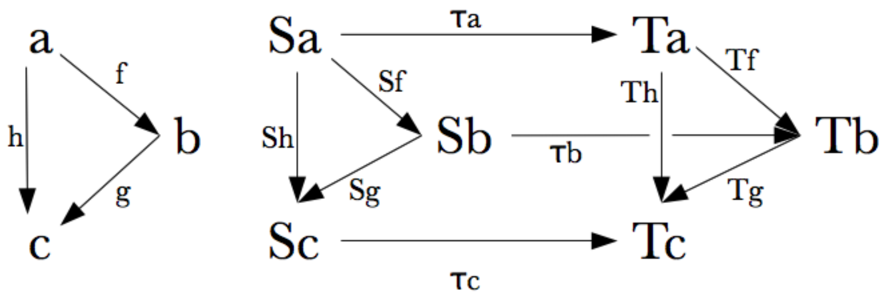
A functor $T : \mathcal{C} \to \mathcal{B}$ is **faithful** when for every pair c, c' of objects in $\mathcal{C}$ and every pair $f_1,\, f_2 : c \to c'$ of (parallel) arrows of $\mathcal{C}$, $Tf_1 = Tf_2$ implies $f_1 = f_2$.

The functor $T : \mathcal{C} \to \mathcal{B}$ is called **full** when for every pair c, c' of objects in $\mathcal{C}$ and every arrow $g : Tc \to Tc'$ of $\mathcal{B}$, there is an arrow $f : c \to c'$ of $\mathcal{C}$ with $g = Tf$.

Definition 17. A **natural transformation** is a morphism between functors. Given two functors $S, T : \mathcal{C} \to \mathcal{B}$, a natural transformation $\tau : S \to T$ is a function which assigns to each object c in $\mathcal{C}$ a morphism $\tau_c : Sc \to Tc$ of $\mathcal{B}$ such that every arrow $f : c \to c'$ of $\mathcal{C}$ gives the following commutative diagram



In other words, if we regard S as giving a picture of $\mathcal{C}$ in $\mathcal{B}$, then a natural transformation is a set of arrows mapping the picture $S(\mathcal{C})$ in $\mathcal{B}$ to the picture $T(\mathcal{C})$ in $\mathcal{B}$, i.e. the following diagram commutes



The arrows τ_a, τ_b, τ_c of $\mathcal{B}$ are called the components of τ.

A **natural isomorphism** $\tau : S \cong T$ between functors $S, T : \mathcal{C} \to \mathcal{B}$ is a natural transformation τ such that every component τ_c is invertible in $\mathcal{B}$. If this is the case, then $(\tau_c)^{-1}$ are the components of a natural isomorphism $\tau^{-1} : T \to S$. With the help of this concept, one can define the equivalence between categories $\mathcal{C}$ and $\mathcal{D}$. This is defined to be a pair of functors $S : \mathcal{C} \to \mathcal{D}$, $T : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $I_{\mathcal{C}} \cong T \circ S$ and $I_{\mathcal{D}} \cong S \circ T$. Notice that in this definition equality is **not** used since S and T do not need to be isomorphisms.

There are some important cases where an object or arrows have specific properties. For example, an arrow $e : a \to b$ is invertible in $\mathcal{C}$ if there is an arrow $e' : a \to b$ with $e'e = 1_a$ and $ee' = 1_b$. If this arrow exists it is unique and it is denoted $e' = e^{-1}$. In this case, the objects a and b are isomorphic in $\mathcal{C}$, $a \cong b$. If every arrow in a category $\mathcal{G}$ is invertible, then $\mathcal{G}$ is called a groupoid and each object x in $\mathcal{G}$ determines a group $Hom_{\mathcal{G}}(x, x)$. An arrow $f : x \to x'$ then establishes a group isomorphism $Hom_{\mathcal{G}}(x, x) \cong Hom_{\mathcal{G}}(x', x')$ by conjugation, i.e. $\forall g \in Hom_{\mathcal{G}}(x, x) : g \mapsto fgf^{-1} \in Hom_{\mathcal{G}}(x', x')$.

2.2.2 Monoidal Categories

The most important concept for us now is the one of monoidal categories, which are basically a category supplied with a "product" $\Box$, e.g. the direct product $\times$, the direct sum $\oplus$ or the tensor product $\otimes$. There are to kinds of this category, depending on the required strength of the associative law. A strict monoidal category $\langle \mathcal{B}, \Box, e \rangle$ is a category $\mathcal{B}$ with a bifunctor $\Box : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ which is **strictly associative**, this means that both functors $\Box \circ (\Box \times 1_{\mathcal{B}})$ and $\Box \circ (1_{\mathcal{B}} \times \Box)$ are **exactly equal**[5], i.e. $(\mathcal{B} \times \mathcal{B}) \times \mathcal{B} = \mathcal{B} \times (\mathcal{B} \times \mathcal{B})$. This category also contains an object e which is a left and right **unit** for $\Box$ such that $\Box \circ (e \times 1_{\mathcal{B}}) = id_{\mathcal{B}} = \Box \circ (1_{\mathcal{B}} \times e)$, where $e \times 1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ is the functor which sends an object c in $\mathcal{B}$ to $\langle e, c \rangle$ [6]. The product $\Box$ assigns to each pair of objects a, b in $\mathcal{B}$ an object $a \Box b$ in $\mathcal{B}$ and to each pair of arrows $f : a \to a'$, $g : b \to b'$ an arrow $f \Box g : a \Box b \to a' \Box b'$ such that $1_a \Box 1_b = 1_{a \Box b}$ and $(f' \Box g') \circ (f \Box g) = (f' \circ f) \Box (g' \circ g)$.

Definition 18. A (relaxed) **monoidal category** $\langle \mathcal{B}, \Box, e, \alpha, \lambda, \rho \rangle$ is a category as the one described above, but with the associativity law only up to a natural isomorphism α and the left and right unit for $\Box$ also up to natural isomorphisms λ and ρ respectively, such that

$$\alpha_{a,b,c} : a \Box (b \Box c) \cong (a \Box b) \Box c \,; \; \lambda_a : e \Box a \cong a \,, \; \rho_a : a \Box e \cong a \; \forall a, b, c \text{ in } \mathcal{B}.$$

[5] Associativity of this bifunctor means that $\Box$ is associative for objects and for arrows.

[6] The unit law states for objects $e \Box c = c = c \Box e$ and for arrows $1_e \Box f = f = f \Box 1_e$.

and the following diagrams commute for all a, b, c, d in $\mathcal{B}$:

$$\begin{tikzcd}
 & (a\square b)\square(c\square d) \arrow[dr, "\alpha"] & \\
a\square(b\square(c\square d)) \arrow[ur, "\alpha"] \arrow[d, "1\square\alpha"'] & & ((a\square b)\square c)\square d \\
a\square((b\square c)\square d) \arrow[rr, "\alpha"'] & & (a\square(b\square c))\square d \arrow[u, "\alpha\square 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a\square(e\square c) \arrow[rr, "\alpha"] \arrow[dr, "1\square\lambda"'] & & (a\square e)\square c \arrow[dl, "\rho\square 1"] \\
 & a\square c &
\end{tikzcd}$$

The identification of multiple products of the category $\mathcal{B}$ is made in a coherent way with the help of the coherence theorem, which states that all diagrams using the natural transformations α, λ, ρ commute (c.f. [24], Sec. VII). This allows us to establish an equivalence to a strict monoidal category and regard the associativity law in a monoidal category as the usual one, but keeping always in mind that we are dealing with equivalences rather than strict equalities. We need this result since the category of representations of a Hopf algebra is not strict, [8]. We can see this by considering the more simple case of representations of the $SU(2)$ group. Recall that the coupling of three angular momenta j_1, j_2, j_3 depends on the way they are coupled, but the resulting vector spaces are isomorphic to each other. We have then $(j_1 \otimes j_2) \otimes j_3 \cong j_1 \otimes (j_2 \otimes j_3)$.

Definition 19. A **category with duals** is a monoidal category $\langle \mathcal{C}, \square, e, \alpha, \lambda, \rho \rangle$ with a functor $* : \mathcal{C}^{op} \to \mathcal{C}$ and the following natural transformations

1. $\tau : 1 \to **$ i.e. $\phi^{**} \cong \phi$, where ϕ is an object or arrow of $\mathcal{C}$.
2. $\gamma : (* \times *) \circ \otimes \to * \circ \otimes^{op}$ i.e. $\phi^* \otimes \psi^* \cong (\psi \otimes \phi)^*$ where ϕ, ψ are objects or arrows of $\mathcal{C}$.
3. $\nu : e \to e^*$, with $e^* \cong e$.

where $\mathcal{C}^{op}$ is the category where the arrows of $\mathcal{C}$ are inverted and the objects are the same. For the natural transformations in 1. and 2. is required each of their components to be isomorphisms.

Remark. A category with strict duals is a category in which τ, γ, and ν are the identity maps, [8].

Before further specification of the categories we are interested in, we give the definition of a monoidal functor, which is a functor $F : \langle \mathcal{B}, \square, e, \alpha, \lambda, \rho \rangle \to \langle \mathcal{V}, \square', e', \alpha', \lambda', \rho' \rangle$ between monoidal categories which respects the monoidal product in the sense that for all objects a, b, c and arrows f, g in $\mathcal{B}$ we have

$$\begin{aligned}
F(a\square b) &= Fa\square' Fb\,, & F(f\square g) &= Ff\square' Fg\,, & Fe &= e'\,; \\
F\alpha_{a,b,c} &= \alpha'_{Fa,Fb,Fc}\,, & F\lambda_a &= \lambda'_{Fa}\,, & F\rho_a &= \rho'_{Fa}.
\end{aligned}$$

2.2.3 Spherical categories

In this section we follow [8, 7] to establish the relation between the graphs and their properties described in Section 1.1 and a special type of monoidal categories, the spherical categories, which are defined to be a pivotal category satisfying the extra condition that the right and left traces of the endomorphisms of all objects are equal. Hence, in spherical categories closed planar graphs are equivalent under isotopies of S^2, while in a pivotal category planar graphs are equivalent under isotopies of a plane. Recall that a sphere can be regarded as a plane with the infinity points identified by the stereographic projection. It is then natural to regard closed graphs on a plane as closed graphs on a sphere, however, the extra condition mentioned is needed to allow us to pass an edge of a closed graph through the point at infinity such that the resulting and the original closed networks represent the same value.

In a more mathematical language, the representations of Hopf algebras with a distinguished element satisfying some conditions, like involutory or ribbon Hopf algebras (cf. Sec. 2.1.2) form in the non-degenerate case spherical categories and the morphisms of these categories are represented by planar graphs. If the category of representations of a Hopf algebra is degenerate, one has to take a non-degenerate quotient category. This quotient, however, is not the category of representations of a finite dimensional Hopf algebra since it is not possible to assign a dimension to each object, i.e. a positive integer which is additive under direct sum and multiplicative under tensor product, [7].

Definition 20. A **pivotal category** is a category $\mathcal{C}$ with duals and a morphism $\epsilon(c) : e \to c \otimes c^*$ for each object c in $\mathcal{C}$ satisfying the following conditions

1. For all $f : a \to b$ in $\mathcal{C}$ following diagram commutes:

$$\begin{tikzcd}
e \arrow[r, "\varepsilon(a)"] \arrow[d, "\varepsilon(b)"'] & a \otimes a^* \arrow[d, "f \otimes 1"] \\
b \otimes b^* \arrow[r, "1 \otimes f^*"'] & b \otimes a^*
\end{tikzcd}$$

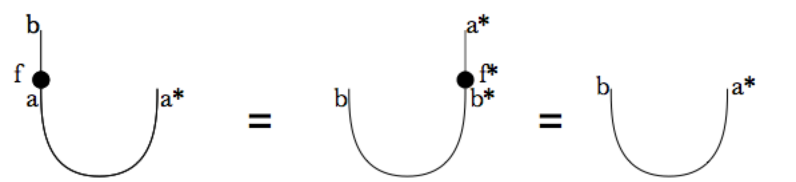
2. For all objects a in $\mathcal{C}$ following composition is 1_{a^*}:

$$\begin{tikzcd}
e \otimes a^* \arrow[r, "\varepsilon(a^*) \otimes 1"] & (a^* \otimes a^{**}) \otimes a^* \arrow[r, "\alpha"] & a^* \otimes (a^{**} \otimes a^*) \arrow[dr, "1 \otimes \gamma"] & \\
& & & a^* \otimes (a \otimes a^*)^* \arrow[d, "1 \otimes \varepsilon(a)^*"] \\
a^* \arrow[uu, "\lambda^{-1}(a^*)"] \arrow[r, "1_{a^*}"'] & a^* & a^* \otimes e \arrow[l, "\rho"] & a^* \otimes e^* \arrow[l, "1 \otimes \nu^{-1}"]
\end{tikzcd}$$

3. For all objects a, b in $\mathcal{C}$ following composite is required to be $\epsilon(a \otimes b)$:

$$\begin{tikzcd}
a\otimes(e\otimes a^*) \arrow[r, "1\otimes(\varepsilon(b)\otimes 1)"] & a\otimes((b\otimes b^*)\otimes a^*) \arrow[r, "1\otimes\alpha"] & a\otimes(b\otimes(b^*\otimes a^*)) \arrow[d, "\alpha^{-1}"] \\
a\otimes a^* \arrow[u, "1\otimes\lambda^{-1}"] & \boxed{e \xrightarrow{\varepsilon(a\otimes b)} (a\otimes b)\otimes(a\otimes b)^*} \arrow[l, "\varepsilon(a)"'] & (a\otimes b)\otimes(b^*\otimes a^*) \arrow[l, "1\otimes\gamma"']
\end{tikzcd}$$

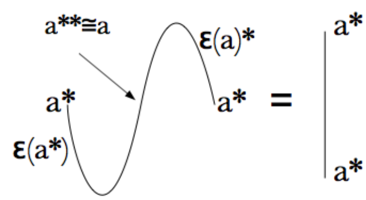
The morphism $\epsilon(c)$ in the above definition corresponds to ${}^{c}\bigcup^{c^*}$. This correspondence together with the identification of the identity 1_a with a straight line $\Big|_a^a$ serves to understand the above commuting diagrams. Notice that the dual f^* of f, according to $f : a \to b$ in $\mathcal{C}$ and the functor $* : \mathcal{C}^{op} \to \mathcal{C}$, is $f^* \equiv (f^{op})^* : b^* \to a^*$ since $f^{op} : b \to a$. Hence we have for the diagrammatic expression of the first condition



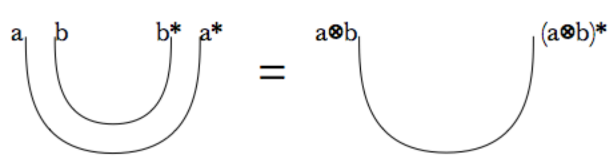
$$(f\otimes 1)\varepsilon(a) = (1\otimes f^*)\varepsilon(b) = \varepsilon' : e \to b\otimes a^*$$

where $\epsilon' : e \to b \otimes a^*$ is a morphism in $\mathcal{C}$.

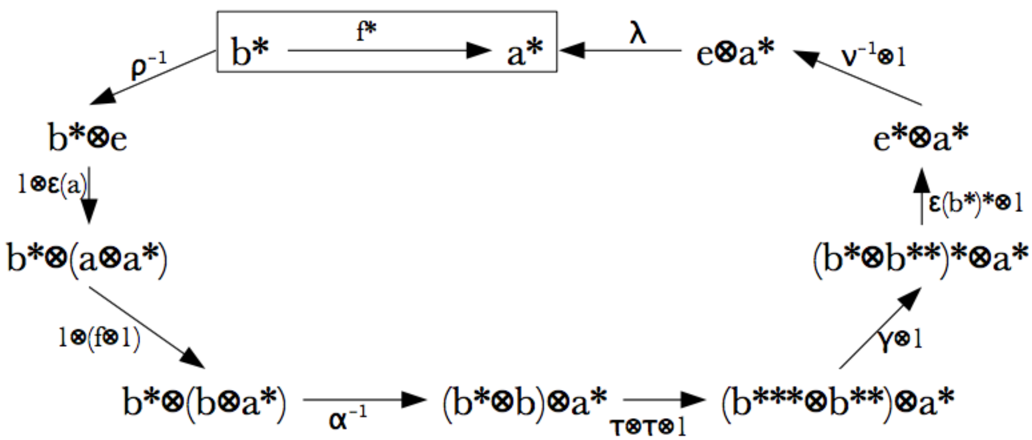
Now, notice that in the second condition of definition 20 the dual of $\epsilon(c)$ is $\epsilon(c)^* : c^{**} \otimes c^* \to e^*$, thus, the resulting diagram corresponding to this property of pivotal categories is



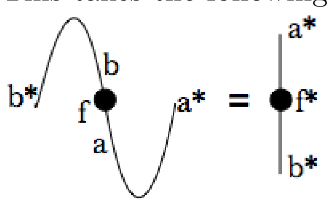
The third condition expresses the compatibility of the pivotal structure with the monoidal product $\otimes$ and the counit e:



Remark. *(i)* In a pivotal category the following composite is the dual $f^* : b^* \to a^*$ of any morphism $f : a \to b$ of $\mathcal{C}$:



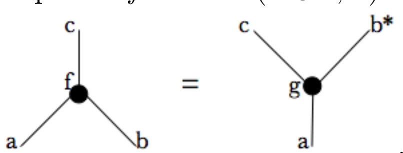
This takes the following diagrammatic form



(ii) In this type of category for all a, b, c in $\mathcal{C}$ there are natural isomorphisms such that

$$Hom(a \otimes b,\, c) \cong Hom(b,\, a^* \otimes c) \quad ; \quad Hom(a,\, b \otimes c) \cong Hom(a \otimes c^*,\, b)$$
$$Hom(a \otimes b,\, c) \cong Hom(a,\, c \otimes b^*) \quad ; \quad Hom(a,\, b \otimes c) \cong Hom(b^* \otimes a,\, c)$$

This natural isomorphisms can be described diagrammatically as following in the case of a morphism $f \in Hom(a \otimes b,\, c) \cong Hom(a,\, c \otimes b^*) \ni g$



This implies that given a trivalent planar graph with an orientation of each edge, a distinguished edge at each vertex, a map from edges to objects and a map from vertices to morphisms, then the graph can be evaluated to obtain a morphism. The fact that this data corresponds to a pivotal category implies that the resulting morphism is dependent only on the isotopy class of the graph, *if* the data is carried along with the isotopy. In Chapter 3 we will see that this is used to construct invariants of 3-manifolds which represent partition functions in analogy to the ones constructed in Section 1.3.

(iii) Main examples of pivotal categories are categories of representations of Hopf algebras which are in general not strict. The difference between a pivotal and *strict* pivotal category is rather technical. Objects that are canonically isomorphic in a pivotal category are equal in a strict pivotal category. We will only consider the latter ones, however, every pivotal category is equivalent to a strict one, hence, there is no loss of generality, [7, 8, 24].

Now, in order to evaluate graphs one needs to define a way of mapping a graph, which represents basically a morphism, into a scalar. More general,[7] one defines a so-called trace map as follows:

[7] See also Section 3.3.

Definition 21. For any object a in a pivotal category $\mathcal{C}$, the monoid[8] $End(a)$ has two **trace maps** $tr_L, tr_R : End(a) \to End(e);\ f \mapsto tr_{L,R}(f)$, which are defined to be the following composites respectively

$$\begin{tikzcd}
 & a^*\otimes a \arrow[r, "\tau\otimes\tau"] & a^{***}\otimes a^{**} \arrow[rr, "\gamma"] & & (a^*\otimes a^{**})^* \arrow[d, "\epsilon(a^*)^*"] \\
a^*\otimes a \arrow[ur, "1\otimes f"] & a^*\otimes a^{**} \arrow[l, "1\otimes\tau^{-1}"'] & e \arrow[l, "\epsilon(a^*)"] \arrow[r, "1_e"'] & e & e^* \arrow[l, "\nu^{-1}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
a\otimes a^* \arrow[r, "\tau\otimes 1"] & a^{**}\otimes a^* \arrow[rr, "\gamma"] & & (a\otimes a^*)^* \arrow[d, "\epsilon(a)^*"] \\
a\otimes a^* \arrow[u, "f\otimes 1"] & e \arrow[l, "\epsilon(a)"] \arrow[r, "1_e"'] & e & e^* \arrow[l, "\nu^{-1}"]
\end{tikzcd}.$$

In a strict pivotal category these definitions simplify to

$$\begin{aligned} tr_L(f) &= \epsilon(a^*)(1\otimes f)\epsilon(a^*)^* \\ tr_R(f) &= \epsilon(a)(f\otimes 1)\epsilon(a)^* \end{aligned}$$

since $a^{**} = a,\ a\otimes a^* = (a\otimes a^*)^*$.

Definition 22. A **spherical category** is a pivotal category for which both trace maps coincide for all objects a in $\mathcal{C}$ and all morphisms $f \in End(a)$ of $\mathcal{C}$, i.e.

$$tr_L(f) = tr_R(f)\,;\ \forall a \,\text{in}\, \mathcal{C},\ \forall f \in End(a).$$

Remark. This is equivalent to $tr_L(f) = tr_L(f^*)\,,\ \forall f \in End(a)$. Also, in a spherical category $tr_L(f\otimes g) = tr_L(f)\cdot tr_L(g)\,,\ \forall f \in End(a),\ \forall g \in End(b)$, where $\cdot$ is the binary operation in $End(e)$.

With the definition of trace and the condition of spherical categories we are able to define for each object c in a spherical category its quantum dimension by $dim_q(c) = tr_L(1_c)$, cf. Section 1.1 and 2.12 on page 55. The spherical condition implies then $dim_q(c) = dim_q(c^*)$.

Now we consider categories of finitely generated modules of spherical Hopf algebras which are additive[9] spherical categories.

Definition 23. A **spherical Hopf algebra** over a ring[10] $\mathbb{F}$ is a Hopf algebra H with an antipode S and an additional structure given by an element $w \in H$ that satisfies following conditions:

[8] A monoid is an algebraic structure with an associative multiplication and an identity element, e.g. a semi-group, a monoidal category with one object, cf. [24].

[9] In this case "additive" means that all Hom-sets are finitely generated abelian groups w.r.t. pointwise addition and that the spherical structure is compatible with the additive structure. We assume as well that $End(e) = \mathbb{F}$ is a field and that each Hom-set is a finite dimensional vector space over $\mathbb{F}$, cf. [8].

[10] In any additive monoidal category $\mathbb{F} \equiv End(e)$ is a commutative ring, cf. [21] as referred to in [8].

1. $S^2(a) = waw^{-1}$, $\forall a \in H$
2. $\Delta w = w \otimes w$, i.e. w is a group-like element.
3. $tr(\theta w^{-1}) = tr(\theta w)$, $\forall \theta \in End_H(V)$; where V is any finitely generated left H-module.

Notice that a Hopf algebra with a group-like element that satisfies the first condition above is spherical if either $w^2 = 1$ or all modules are isomorphic to their dual, since the first case is trivial and in the latter case we have from the spherical condition $tr_L(f) = tr_L(f^*)$, thus $tr(\theta w) = tr((\theta w)^*) = tr(w^* \theta^*) = tr(\theta w^{-1})$ by the cyclic property of the trace and the fact that the duals are given by the antipode of H[11].

J. Barrett and B. Westbury proved in [8], that if H is a spherical Hopf algebra over $\mathbb{F}$, then the category of left finitely generated H-modules, which are free[12] as $\mathbb{F}$-modules, is a spherical category since an element w in a Hopf algebra satisfying the first and second condition above determines a pivotal structure for the category of modules, which as we saw in example 12 on page 34 is a monoidal category and the two trace maps are given by $tr_L(\theta) = tr(\theta w)$ and $tr_R(\theta) = tr(\theta w^{-1})$.

If we assign trivalent planar graphs to categories, it is the structure just discussed which allows us to evaluate the graphs to give morphisms depending only on the isotopy class of the graph. Furthermore, spherical categories determine an invariant of isotopy classes of closed graphs embedded on the sphere.

Finally, we describe some conditions needed in order to obtain a spherical category from which one may construct an invariant of (closed) 3-manifolds. These conditions allow to construct an invariant from a finite summation over some objects of the spherical category with the trace of some map as summands, cf. Chapter 3.

The first condition is non-degenerancy which allows the construction of the second condition, which is that of a spherical category being semisimple. This allows an isomorphism between a sum of tensor products of vector spaces given by the Hom-sets and its corresponding Hom-set with a vector space structure, see (2.7).

Following definitions are given in [7],

Definition 24. For any two objects a, b in $\mathcal{C}$, there is a **bilinear pairing**

$$\Theta : \; Hom(a,b) \otimes Hom(b,a) \to \mathbb{F}$$

defined by $\Theta(f,g) = tr_L(fg) = tr_L(gf)$. An additive spherical category is non-degenerate if, for all objects a and b, the pairing Θ is non-degenerate in the usual sense.

A **semisimple** spherical category $\mathcal{C}$ is *additive and non-degenerate* such that there exists a set of inequivalent non-zero[13] objects of a set J, so that for any two objects x, y in $\mathcal{C}$, the natural map given by

$$\bigoplus_{a \in J} Hom(x,a) \otimes Hom(a,y) \to Hom(x,y) \tag{2.7}$$

[11]One can see this by regarding the category with only one object, the Hopf algebra itself, and noticing that the conditions in the definition of a category with duals allow the correspondence between the dualisation and the antipode.

[12]Here, "free" means in this context that the H-module is generated by a finite linearly independent (over $\mathbb{F}$) set of elements of the H-module. In other words the H-module has a basis, cf. [37].

[13]An object a is non-zero if $End(a) \neq 0$.

is an isomorphism. An object is called simple if its endomorphism ring is isomorphic to $\mathbb{F}$.

The set J is fixed by the category since there is an isomorphism between any simple object in the category and its corresponding (unique) element in J. Thus, every element in J is simple. A semisimple spherical category is called finite if J is finite, i.e. if the set of isomorphism classes of simple objects is finite. The dimension of this finite category is then defined by $K = \sum_{a \in J} \dim_q^2(a)$.

Next theorem ensures that it is always possible to construct a non-degenerate additive spherical category from a category which is only additive and spherical. The proof can be found in [8].

Theorem 25. *For a given additive spherical category $\mathcal{C}$, the additive subcategory $\mathcal{J}$ defined to have the same set of objects and the Hom-sets defined by*

$$Hom_{\mathcal{J}}(a, b) = \{f \in Hom_{\mathcal{C}}(a, b)\,;\, \Theta(f, g) = tr_L(fg) = 0 \text{ for all } g \in Hom_{\mathcal{C}}(b, a)\}$$

gives a quotient $\mathcal{C}/\mathcal{J}$ which is a non-degenerate additive spherical category.

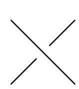
The above theorem is very important since this construction is the general way of attaining the semisimple condition needed for the construction of invariants of closed 3-manifolds. This result is proved in [8] and states that, given a spherical Hopf algebra H over a field $\mathbb{F}$, the non-degenerate quotient of the spherical category of finitely generated left H-modules is semisimple.

We will see in Chapter 3 that the above gives rise to the generalized construction of the Turaev-Viro invariant as a state sum over the quotient of the category of representations of the deformed quantum enveloping Hopf algebra $U_q(\mathfrak{sl}_2)$, which can be made a spherical Hopf algebra. Hence, these categories can be seen as the generalization of the objects introduced in Section 1.1.

2.3 Some More Diagrams: The Temperley-Lieb Recoupling Theory

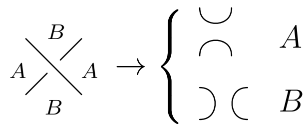
In this section, the diagrams of the previous sections are formalized and it is intended as a short introduction in this very broad area of knot theory. We follow [19] for the definition of the bracket polynomial and a discussion of the Jones polynomial, as well as [20] for the discussion on Temperley-Lieb recoupling theory.

First we introduce the bracket polynomial to associate a knot to an invariant in order to evaluate the given diagram, which is defined as an schematized picture of the given knot in a plane. The diagram is composed of curves that cross in 4-valent vertices. Each of these vertices are equipped with the extra structure given by an under- or over-crossing:

 and 

Overcrossing *Undercrossing*

Now, let us consider a crossing in an unoriented diagram, then two associated diagrams can be obtained by splicing the crossing in two ways as



where A or B denote the type of splitting, i.e. an $A-/B-$split joints the regions labelled A/B at the crossing, [19]. With this convention a split crossing labelled A or B in a diagram can be reconstructed to form the original crossing. Hence, by keeping track of the labelling one can reconstruct the original link from *any* of its descendants. The primitive descendants of a link K, those which have no crossings left, are collections of Jordan curves in the plane[14] and are called the **states of** K. The labelling of the above mentioned splitting process makes the algorithm to evaluate a link unambiguous since each primitive descendant can be associated in the same way to its ancestral link. From these states we are able to construct invariants of knots by averaging over them.

Definition 26. The **bracket polynomial** is defined to be the following formula

$$\langle K \rangle = \sum_{\sigma} \langle K \,|\sigma\rangle \, d^{\|\sigma\|} \tag{2.8}$$

where σ denotes a state of K, $\|\sigma\| = (\text{loops in } \sigma) - 1$ and $\langle K\,|\sigma\rangle$ the product of (commutative) labels of σ, i.e. the product of A's and B's labelling a given state. Here d and the labels are commuting algebraic variables. Hence, in the case relevant for us where $B = A^{-1}$ and $d = -A^2 - A^{-2}$ we have that $\langle K \rangle \in \mathbb{Z}\big[A,\, A^{-1}\big]$ is a polynomial in A and A^{-1} with integer coefficients.

As defined, the bracket polynomial is not a topological invariant, since in this form its value changes under Reidemeister moves. To see this notice that for an over-crossing $\times$ we have[15]

$$\Big\langle \times \Big\rangle = A \Big\langle \asymp \Big\rangle + B \Big\langle \,)\,(\, \Big\rangle \tag{2.9}$$

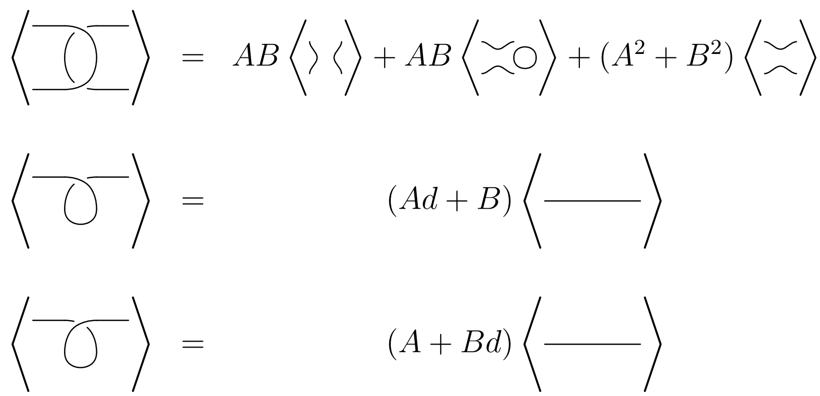
The reiterated use of the above relation[16] is all is needed to compute the bracket. Now, it is easy to see that the diagrams in the first and second Reidemeister moves give the following

[14]Jordan curves are closed loops in the plane homeomorphic to S^1.

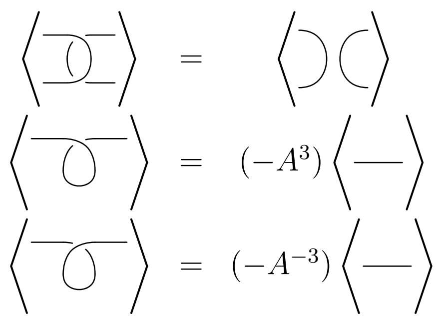
[15]This is understood as regarding the over-crossing and the splits as part of a bigger diagram, i.e. we consider the splitting process locally and the rest of the diagram stays unchanged. For the proof of this relation and the following ones see [19, Part I, Sec. 3].

[16]Its analog one for the under-crossing is not needed since switching the crossings exchanges the roles of A and B.

bracket relations



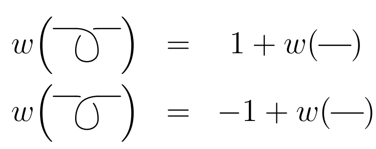
So, if $B = A^{-1}$ and $d = -(A^2 + A^{-2})$, then we obtain



Note that from the definition of the bracket polynomial, it follows that $\langle OK \rangle = d \langle K \rangle$ where OK denotes the disjoint union of a closed loop to the diagram K. From this, it follows that the bracket with the conditions $B = A^{-1}$, $d = -(A^2 + A^{-2})$ is an invariant of regular isotopy, i.e. under the moves II and III[17]. In order to have an invariant of ambient isotopy, i.e. under moves I, II, III, we need to normalize the bracket polynomial. The normalized bracket $\mathcal{L}_K$ for oriented links K is defined by

$$\mathcal{L}_K = (-A^3)^{-w(K)} \langle K \rangle \tag{2.10}$$

where $w(K)$ is the writhe of K defined by $w(K) = \sum_p \epsilon(p)$ where p runs over all crossings in K and $\epsilon(p) = \pm 1$ is the sign of the over- and under-crossing respectively. Hence, (2.10) is an invariant of ambient isotopy since $w(K)$ is an invariant of regular isotopy and



Remark. The mirror image K^* of a link K is the link resulting from the interchange of over- and under-crossings. Let K^* be the mirror image of an oriented link K, then

$$\begin{aligned} \langle K^* \rangle (A) &= \langle K \rangle (A^{-1}) \\ \mathcal{L}_{K^*}(A) &= \mathcal{L}_K(A^{-1}) \end{aligned}$$

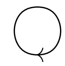
Hence, if $\mathcal{L}_K(A) \neq \mathcal{L}_K(A^{-1})$, then K is not ambient isotopic to its mirror image K^*.

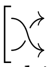
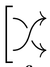
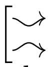
[17] The invariance under move III follows from the invariance under move II, [19].

We now give the definition of the Jones polynomial as well as its relation to the normalized bracket polynomial, which ensures the existence and well-definiteness of the one-variable Jones polynomial.

Definition 27. The one-variable Jones polynomial $V_K(t)$ is a polynomial in $t^{1/2}$ with finitely many positive and negative powers of $t^{1/2}$ (i.e. a Laurent polynomial) associated to an oriented link K and which satisfies the following properties:

1. If K is ambient isotopic to K', $V_K(t) = V_{K'}(t)$.

2. $V_{O^+}(t) = 1$, where O^+ denotes the loop having clockwise orientation: 

3. $t^{-1}V_t\Big[$  $\Big] - tV_t\Big[$  $\Big] = \left(t^{1/2} - t^{-1/2}\right)V_t\Big[$  $\Big]$, where the brackets are a notation for links which differ from each other only in the showed crossing.

With this definition of $V_K(t)$ and (2.10), we have that the normalized bracket relates to the 1-variable Jones polynomial as

$$\mathcal{L}_K(t^{-1/4}) = V_K(t).$$

This shows that even if the definition above is not obviously well-defined we can take it as given since $\mathcal{L}_K$ exists and is well-defined, [19].

The Jones polynomial has been generalized in many ways. One way involves choosing a compact Lie group G and an irreducible representation. Then a polynomial invariant of oriented links corresponding to these irreducible representations is constructed by using solutions of the Yang-Baxter equations (2.3). With this method, the original Jones polynomial corresponds to choosing $G = SU(2)$ with its standard representation on $\mathbb{C}^2$, [5]. In fact, the quantum group $U_q(\mathfrak{sl}_2)$ gives rise to the Jones polynomial and the Yang-Baxter equations are the algebraical form of Move III in Section 1.1, [36].

2.3.1 Temperley-Lieb Algebra

Now we proceed with the introduction[18] of an algebra which allows us to construct the bracket polynomial and the projectors given in definition 5 on page 15. For this, first define the elementary tangles $U_0 = 1_n, U_1, U_2, \ldots, U_{n-1} \in T_n$ of the n-strand algebra T_n. We can think of each of these tangles as being n strands fixed at their ends in a box having n fixing points at the upper margin (outputs) and n fixing points at the lower margin (inputs), such that in U_i the k^{th} output is connected for $k \neq i, i+1$ to the k^{th} input and the i^{th} input and output are connected to the $(i+1)-th$ input and output respectively. The multiplication of the elements is given by attaching the output with the input, i.e. by stacking the boxes one above the other. The algebra is given by the following defining relations:

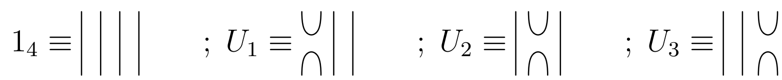
1. $U_i^2 = dU_i$, where d is a value assigned to a closed loop.

2. $U_iU_{i+1}U_i = U_i$

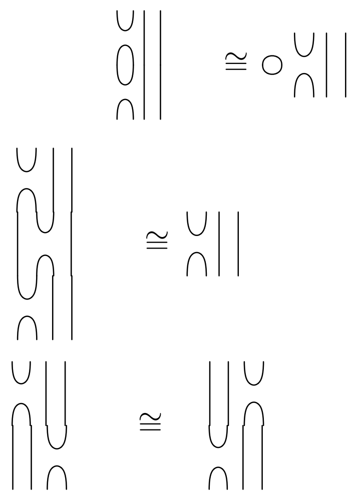
[18] The following can be found in [20].

3. $U_iU_j = U_jU_i$ for $|i-j| > 1$.

Example. Consider the case $n = 4$. The elementary tangles in T_4 are given by



with the relations, say



Where $\simeq$ means that the tangles are equivalent. This is the case when they are regularly isotopic relative to their (fixed) endpoints. Every planar non-intersecting n-tangle is equivalent to a product of the n elementary tangles and two such products represent equivalent tangles if and only if one product can be obtain from the other by the relations above.

Definition 28. The **Temperley-Lieb algebra** T_n is a free additive algebra over the set of rational functions with numerator and denominator in $\mathbb{Z}\big[A,\, A^{-1}\big]$ and with multiplicative generators $1_n, U_1, \ldots, U_{n-1}$ and the relations given above. The value of the loop is $d = -A^2 - A^{-2}$ and since A and A^{-1} commute with all elements of T_n, thus d too.

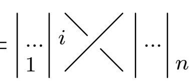
In order to evaluate an n-tangle x we define a trace map $tr : T_n \to \mathbb{Z}\big[A,\, A^{-1}\big]$ defined by $tr(x) = \langle \bar{x} \rangle$ and $tr(x+y) = tr(x) + tr(y)$ where $\bar{x}$ denotes the standard closure of x obtained by joining the k^{th} input and output from outside the tangle, such that each strand forms a loop going from the bottom of the n-tangle to the top of the n-tangle. This trace map is defined also for the generalization of T_n to the tangle algebra generated multiplicatively by n-strand tangles of general form, e.g. there can be crossings inside the box representing the tangle. Note that the cyclic property of the trace map, i.e. $tr(ab) = tr(ba)$, is a direct consequence of the properties of the bracket polynomial[19] and the standard closure:

$$
\begin{aligned}
tr(ab) &= \langle \overline{ab} \rangle = \sum_{\sigma} \langle \overline{ab}|\sigma \rangle d^{\|\sigma\|} = \sum_{\sigma} \left\langle \bar{a}|\sigma \right\rangle \left\langle \bar{b}|\sigma \right\rangle d^{\|\sigma\|} \\
&= \sum_{\sigma} \left\langle \bar{b}|\sigma \right\rangle \left\langle \bar{a}|\sigma \right\rangle d^{\|\sigma\|} = \sum_{\sigma} \langle \overline{ba}|\sigma \rangle d^{\|\sigma\|} = tr(ba)
\end{aligned}
$$

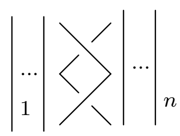
[19]Recall that the coefficients $\left\langle \bar{ab}\,|K \right\rangle$ corresponding to the states of K are products of commutating factors.

Remark. In the case where $x \in T_n$, i.e. x is a product of U_i's, we have that $\bar{x}$ is a disjoint union of Jordan curves so $tr(x) = d^{\|\bar{x}\|}$, where $\|\bar{x}\|$ is the number of loops in the plane. Thus, each product of U_i's correspond to a single bracket state.

Consider now the Artin braid group B_n which is one special case of the above generalization and is useful to formalize the concept of n-edge in Section 1.1. As the name suggests a braid in B_n is a collection of n strands woven into a single n-tangle. Notice that weaving two strands, say the i^{th} and $(i+1)^{th}$, is nothing more than crossing one above the other. There are, however, two ways of doing this, weaving the i^{th} over the $(i+1)^{th}$ forming an over-crossing or vice versa, forming an under-crossing. Denote these two ways, σ_i and σ_i^{-1} respectively. Denoting σ_i^{-1} as the inverse of σ_i makes sense since they are, in fact, inverse to each other:

$\sigma_i =$  and $\sigma_i^{-1} =$ 

such that

$\sigma_i \sigma_i^{-1} =$  $\cong 1_n$

due to the second Reidemeister move.

Now, considering (2.9) each σ_i has an horizontal type A smoothering $H(\sigma_i) = U_i$ and a vertical type A^{-1} smoothering $V(\sigma_i) = 1_n$ and for σ_i^{-1} the opposite types. Hence, each bracket state of the closure $\bar{b}$ of a braid b corresponds to the closure of an element of the Temperley-Lieb algebra. This means that there exists a representation $\rho : B_n \to T_n$ given by

$$\begin{aligned} \rho(\sigma_i) &= AU_i + A^{-1}1_n \\ \rho(\sigma_i^{-1}) &= A^{-1}U_i + A1_n \end{aligned}$$

which allows us to compute $\langle \bar{b} \rangle$ as a sum of trace evaluations of elements of T_n, i.e. $tr(\rho(b)) = \langle \bar{b} \rangle$. Hence, the n-edges introduced in Section 1.1 are related to the Temperley-Lieb algebra via this representation and, moreover, its value defined as the value of the bracket polynomial (2.8) of its closure is well-defined and computable via the above representation.

Before discussing some properties of the projectors, or n-edges, and the 3-vertex as defined in Section 1.1, we give a more general and formal definition of the projectors as a sum of n-tangles.

First, consider the element $f_i \in T_n$ defined inductively for $i = 0, 1, \ldots, n-1$ by

$$\begin{aligned} (i) \quad f_0 &= 1_n \\ (ii) \; f_{i+1} &= f_i - \mu_{i+1} f_i U_{i+1} f_i \end{aligned}$$

where $\mu_1 = d^{-1}$, $\mu_{i+1} = (d - \mu_i)^{-1}$.

With this definition the elements f_i have following properties for $i \in \{0, 1, \ldots, n-1\}$, [20]:

1. $f_i^2 = f_i$ which corresponds to the projection property in Section 1.1

2. $f_i U_j = U_j f_i = 0$ for $j \leq i$, corresponds to the irreducibility of an i-edge as in Section 1.1

3. $tr(f_{n-1}) = \Delta_n\big(-A^2\big)$ and $\mu_{i+1} = \Delta_i/\Delta_{i+1}$ with $\Delta_0 = 1$ and

$$\Delta_n(x) = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}$$

   is called the n^{th} Chebyschev polynomial. Furthermore, it holds that $tr(f_{i-1}) = \Delta_i$ when the trace is taken with respect to i-tangles since $T_1 < T_2 < \cdots < T_n < \ldots$

With respect to the algebra T_n there is a unique non-zero element given by $f_{n-1} \in T_n$ such that the second property holds for all $j = 1, \ldots, n-1$. Hence, with the definition of the projector $\boxed{n}$ given below, we have[20] $f_{n-1} = \boxed{n}$.

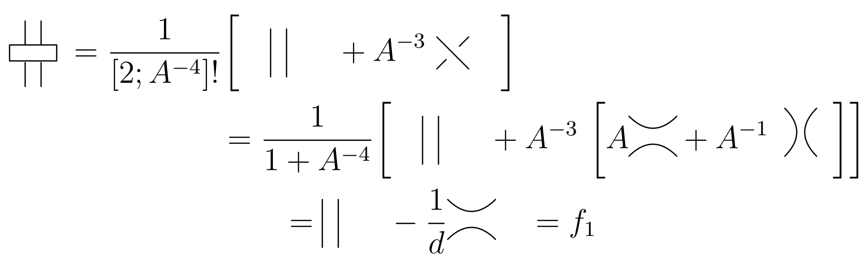
Definition 29. For a given positive integer n, define the n-tangle obtained from the sum of elements of the minimal representations of the symmetric group S_n, as a product of transpositions, as follows:

$$\boxed{n} = \frac{1}{[n; A^{-4}]!} \sum_{\sigma \in S_n} \big(A^{-3}\big)^{\|\sigma\|} \boxed{\hat{\sigma}}$$

where $\|\sigma\| \in \mathbb{N}$ is the number of transpositions in the minimal representation of σ and $[n; A^{-4}]! = \sum_{\sigma \in S_n} \big(A^{-4}\big)^{\|\sigma\|} = \prod_{k=1}^{n} \frac{1-A^{-4k}}{1-A^{-4}}$ is the q-deformed factorial with $q = A^2$ and the property that for $A = \pm 1$ we have $[n]! = n!$. Notice that $\sigma \in S_n$ is represented by an element $\hat{\sigma} \in B_n$.

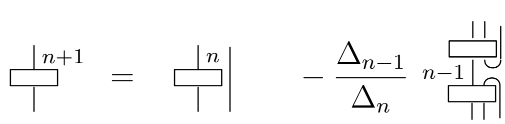
This is a more general definition as the one given in Section 1.1. Here the braiding of the n strands also correspond to permutations, but the concept has now being formalized by identifying a transposition $(i\, i+1) \in S_n$ with the generator σ_i of the Artin braid group B_n.

Example. In the case where $n = 2$ the explicit expansion of the 2-edge is as follows:



where $d = -A^2 - A^{-2}$.

With the diagrammatic notation, the recursion relation defining $f_{n-1} = \boxed{n}$ is given by


(2.11)

[20] Since any n-tangle can be seen as a sum of elements in the Temperley-Lieb algebra we can regard $\boxed{n}$ as in T_n.

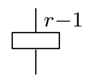
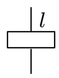
from which the recursion relation for the evaluation of its closure follows, cf. Footnote 16 on page 16:

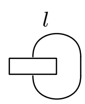
$$\Delta_{n+1} = \Delta_n d - \Delta_{n-1}$$

Both relations above are very useful for calculations of closed diagrams, for instance, the theta-value of a trivalent vertex discussed at the end of this section.

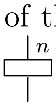
Finally we discuss some properties of the projectors and 3-vertices which are important for the general discussion of the recoupling theory and the q-deformed $6j$-symbols. In this case we impose $A^2 = q$ where q is a $2r$-th primitive root of unity such that $q^r = -1$, $q = e^{i\pi/r}$ and $d = -q - q^{-1}$. Thus, the trace evaluation of the element f_{r-2}, given by

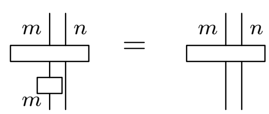
$$tr(f_{r-2}) = \Delta_{r-1} = (-1)^{r-1}\frac{q^r - q^{-r}}{q - q^{-1}},$$

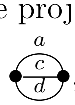
vanishes. In fact, the stronger[21] identity  $= 0$ holds, [20]. Furthermore, since $q = e^{i\pi/r}$ the trace of  does not vanish for $0 \leq l \leq r-2$ and in this range we have

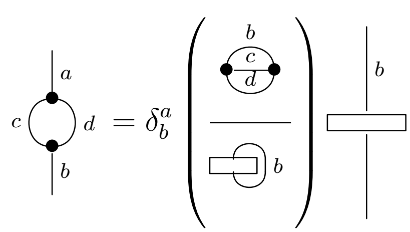
$\Delta_l =$  $= (-1)^l \dfrac{\sin(\pi(l+1)/r)}{\sin(\pi/r)}.$ (2.12)

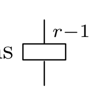
Using l'Hôpital's rule we see that the above expression delivers the loop value as in Section 1.1, where $r \to \infty$ and the n-edges are representations of $SU(2)$.

For generic d we remark that the coefficients of left-right symmetric terms in the expansion of the projectors are equal. In addition, one useful expression of the projection is given by  $= 1_n + \mathcal{U}_n$ where $\mathcal{U}_n$ is a sum of products of the generators of T_n, hence, it has strands turning back. From this and the irreducibility of projectors we obtain the following identity

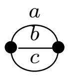


Now, consider two trivalent vertices (a, d, c) and (c, d, b) joint together at the edges c and d. An important property of this object -useful to prove the orthogonality and the Biedenharn-Elliott identity of $q-6j$-symbols and their relation to tetrahedra- is the following relation to the projector. Assume that $\Delta_a \neq 0$ and denote the theta-evaluation of the 3-vertex (a, c, d) by , then

 (2.13)

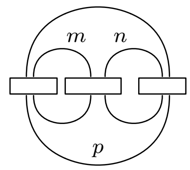
[21] Stronger in the sense that if any closed network contains , then its evaluation vanishes.

Hence, it vanishes whenever $a \neq b$.

To conclude with this section, we mention shortly[22] the evaluation of the theta-net $\theta(a,\, b,\, c) =$ 

in the Temperley-Lieb algebra for the case $q = e^{i\pi/r}$. First, notice that the theta-function can be written in terms of three projectors as follows

$$\theta(a,\, b,\, c) = \tag{2.14}$$



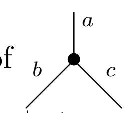
where

$$m = \frac{a+b-c}{2},\ n = \frac{b+c-a}{2},\ p = \frac{c+a-b}{2}.$$

Hence, with the help of relation (2.11) one can find, after a tedious calculation, a general formula for the evaluation of a trivalent vertex given by

$$\theta(a,\, b,\, c) = (-1)^{m+n+p} \frac{[m+n+p+1]![n]![m]![p]!}{[n+p]![m+p]![m+n]!}$$

with m, n, p given above and $[n] = (-1)^{n-1}\Delta_{n-1},\ [n]! = [1][2]\ldots[n]$.

An immediate consequence of the above formula is that the evaluation of 

where $a, b, c \leq r-1$ but $a+b+c \geq 2(r-1)$ vanishes since $[m+n+p+1] = (-1)^{m+n+p}\Delta_{m+n+p} = 0$ whenever $m+n+p = r-1$ and hence the quantum factorial vanishes for higher values of this sum.

2.3.2 Recoupling Theory

In this section we discuss one of the most important theorems presented here which allows the proof of the orthogonality relation and the Biedenharn-Elliott identity of the (q-deformed) $6j$-symbols. We discuss the case where $q = e^{i\pi/r}$.

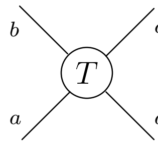
Definition 30. A triple of non-negative integers (a, b, c) is called q-admissible if

$$\begin{array}{cc} (i) & a+b+c \equiv 0\, mod 2 \\ (ii) & b+c-a \geq 0,\ a+b-c \geq 0,\ c+a-b \geq 0 \\ (iii) & a+b+c \leq 2r-4 \end{array} \tag{2.15}$$

The set of q-admissible triples is denoted by ADM_q.

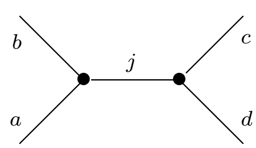
[22]For the derivation of the following formulas and a more detailed discussion of the theta-evaluations see [20].

Consider the set $\mathcal{T}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ of all tangles T of the form



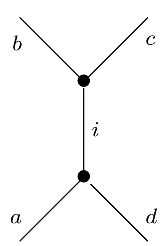
We can regard this tangle as a functional on tangles T' dual to it, cf. Section 1.1. In doing so, we obtain an inner-product $\langle \cdot, \cdot \rangle : \mathcal{T}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \times \mathcal{T}'\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \to \mathbb{C}$, which allows us to define the concept of equality of elements in $\mathcal{T}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, i.e. two tangles are equal if they are equal as functionals on the dual tangles in $\mathcal{T}'\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, in other words, if the inner-product gives the same result for all $T' \in \mathcal{T}'\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$.

Now, the addition of tangles makes $\mathcal{T}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ into a vector space over $\mathbb{C}$ where the set of tangles of the form

$T_j =$  with (a, b, j), $(c, d, j) \in ADM_q$

is a basis[23] for $\mathcal{T}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$.

Similarly, the tangles of the form

$\tilde{T}_i =$  with (b, c, i), $(a, d, i) \in ADM_q$

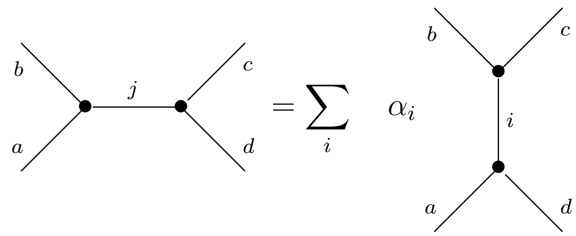
form also a basis for $\mathcal{T}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$.

The recoupling theorem is then a statement about the change of basis:

Theorem 31. *The Recoupling Theorem:*
Let (a, b, j), $(c, d, j) \in ADM_q$, *then there exists* *unique* *real numbers* α_i, $(0 \leq i \leq r-2)$,

[23]The linear independence comes from using equation (2.13) twice.

such that



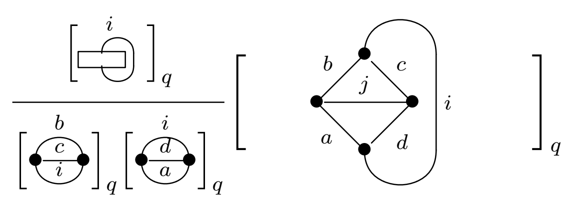
(2.16)

where the sum goes over all non-negative integers i such that (a, d, i), $(b, c, i) \in ADM_q$ and all networks are evaluated at $q = e^{i\pi/r}$. Furthermore, the coefficients $\alpha_i = \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}_q$ are the quantum q-6j-symbols, [20].

Remark. (i) Note that the meaning of this equality is that both sides are interchangeable in all bracket evaluations having these sums inside larger networks. *(ii)* For general value of q the summation is over all admissible triples, i.e. the third condition in (2.15) is not needed.

Next, we give the diagrammatic form of the $q - 6j$-symbols in order to appreciate finally its connection with the tetrahedron which, as seen in Section 1.3, is an important object to build invariants such as the combinatorial analogous of the "path integral" (1.12) over geometries with the exponential of the Hilbert-Einstein action as integrand, cf. Chapter 3. This connection is a direct consequence of the recoupling theorem and the relation (2.13). The formula holds under conditions *(i) and (ii)* in (2.15) for generic q. The $q - 6j$-symbols defined via the above theorem are given[24] by

$$\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}_q =$$



(2.17)

where $[\dots]_q$ denotes the evaluation of the diagram at a given q.

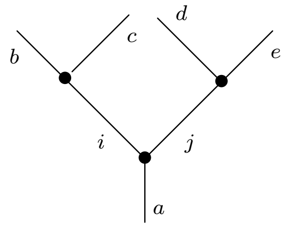
Now, regard the labels a, b, c, d, j as parameters of a function $\{\dots\}_q : \{0, 1, \dots, r-2\}^6 \to \mathbb{R}$ given by the coefficients above, then from a double use of the recoupling theorem we obtain the orthogonality relation

$$\sum_{i=0}^{r-2} \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}_q \left\{ \begin{array}{ccc} d & a & k \\ b & c & i \end{array} \right\}_q = \delta_j^k. \tag{2.18}$$

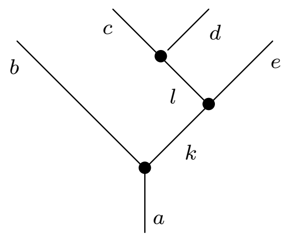
Finally we present an important identity called the Biedenharn-Elliott identity, sometimes

[24]Notice that Moussouris definition of the symbols in [28] is different. For this reason in section 4.2.1 the $6j$-symbols are defined without the loop value and with the theta-net value set to 1, hence, the identity given by the Recoupling theorem is slightly different, namely, the recoupling coefficients are given by the loop value and the evaluation of the tetrahedral network. Because of the different definitions in the literature, extra care in the use of identities is needed when evaluating spin networks.

also called the pentagon identity. Consider the following diagram



expressed in terms of



There are two ways of doing this, one by two consecutive applications of the recoupling theorem, the other one by three applications. Since both ways must give the same result, we obtain the following relation

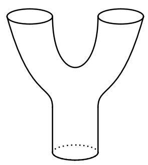
$$\sum_{m=0}^{r-2} \left\{ \begin{array}{ccc} a & i & m \\ d & e & j \end{array} \right\}_q \left\{ \begin{array}{ccc} b & c & l \\ d & m & i \end{array} \right\}_q \left\{ \begin{array}{ccc} b & l & k \\ e & a & m \end{array} \right\}_q = \left\{ \begin{array}{ccc} b & c & k \\ j & a & i \end{array} \right\}_q \left\{ \begin{array}{ccc} k & e & l \\ d & c & j \end{array} \right\}_q . \tag{2.19}$$

These two last properties are very important for the 3-manifold invariants discussed in Chapter 3 and have several consequences as seen earlier. Before coming to the mentioned invariants, we will first discuss briefly other concepts related to the invariants of 3-manifolds, namely, the topological quantum field theory.

2.4 Atiyah's Axiomatic Topological Quantum Field Theory

In [4] M. Atiyah gave an axiomatic approach to topological quantum field theory (TQFT), which will be discussed briefly in this section. The TQFT described in this section is only defined for manifolds with fixed dimension using the concept of cobordisms[25] as morphisms "propagating" a manifold to another manifold of the same dimension but possibly with a different topology. For instance, the figure below shows a cobordism $W = (M; i_+, i_-)$ with $i_+ : S^1 \to \partial M$ and $i_- : S^1 \cup S^1 \to \partial M$, where $\cup$ denotes the disjoint union of two copies of S^1.

[25] A cobordism $W = (M; F_+, F_-; i_+, i_-)$ between d-dimensional manifolds F_+ and F_- is a $(d+1)$-dimensional compact manifold M, such that $i_+ : F_+ \to \partial M$ and $i_- : F_- \to \partial M$ are embeddings with $\partial M = i_+(F_+) \cup i_-(F_-)$ and $i_+(F_+) \cap i_-(F_-) = \emptyset$. If we regard closed manifolds as objects in a category, then the cobordisms can be considered as the morphisms of this category, the **category of cobordisms**, [39].



Definition 32. A **topological quantum field theory** in dimension d defined over a ring Λ consists of the following data:

- To each oriented closed smooth d-dimensional manifold Σ we associate a finitely generated Λ-module $Z(\Sigma)$.
- To each oriented smooth $(d+1)$-dimensional manifold M with boundary we associate an element $Z(M) \in Z(\partial M)$.

subject to the following axioms[26]

1. Z is a functor from the category of compact oriented smooth manifolds, with orientation preserving diffeomorphisms as arrows, to the category of Λ-modules. Another way of defining this functor is from the category of cobordisms to the category of Λ-modules, as in Chapter 3.
2. Involutory: $Z(\Sigma^*) = Z(\Sigma)^*$, where Σ^* denotes Σ with opposite orientation and $Z(\Sigma)^*$ is the dual space.
3. Multiplicativity: $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$ where $\cup$ is the disjoint union.
4. Associativity: For a composite cobordism $M = M_1 \cup_{\Sigma_3} M_2$, see figure below, we have
$$Z(M) = Z(M_2)Z(M_1) \in Hom(Z(\Sigma_1), Z(\Sigma_2)) \tag{2.20}$$
where $\cup_{\Sigma_3}$ denotes the union of two manifolds with a common component Σ_3 of their boundary, i.e. $\Sigma_3 \subseteq \partial M_1$ and $\Sigma_3^* \subseteq \partial M_2$.
5. Non-triviality axioms[27]: $Z(\emptyset) = \Lambda$ and $Z(\Sigma \times \mathbb{I}) = id_{Z(\Sigma)}$ is the identity endomorphism of $Z(\Sigma)$.

The first axiom states that if $f : \Sigma \to \Sigma'$ is an orientation preserving diffeomorphism, i.e. $f \in Diff^+(\Sigma, \Sigma')$, then f induces an isomorphism $Z(f) : Z(\Sigma) \to Z(\Sigma')$ and for $g : \Sigma' \to \Sigma''$, $Z(g \circ f) = Z(g) \circ Z(f)$. Moreover, if f extends to an orientation preserving diffeomorphism f^* from M to M', then $Z(f^*) : Z(M) \mapsto Z(M')$. Notice that $Z(f^*)$ maps the element $Z(M)$ associated to M to an element $Z(M')$ associated to M'. Another way of looking at this is by regarding the category of cobordisms with objects closed manifolds and morphisms cobordisms and the category of Λ-modules with homomorphisms. In this case the

[26]These axioms, excluding the first, are taken as in [5].

[27]Note that if $\Sigma = \emptyset$, then the vector space associated to it is idempotent, i.e. $Z(\emptyset) = Z(\emptyset \cup \emptyset) = Z(\emptyset) \otimes Z(\emptyset)$, thus it is zero or canonically isomorphic to Λ. For similar reasons, if $M = \emptyset$ we have for the $(d+1)$-dimensional empty manifold $Z(\emptyset_{d+1}) = 1$.

cobordism M is associated to a homomorphism $Z(M) \in Z(\partial M)$. Notice that if $\partial M = \Sigma_1 \cup \Sigma_2$ the morphism is, in fact, a homomorphism from $Z(\Sigma_1)$ to $Z(\Sigma_2)$. For instance, if $\Sigma_1 = \Sigma_2$, then $Z(M) \in Z(\Sigma_1)$ and $Z(M)$ is an endomorphism by the action of this element on the module.

When Λ is a field, $Z(\Sigma)$ and $Z(\Sigma)^*$ are dual vector spaces. This case is the most important for physical examples with $\Lambda = \mathbb{C}$, $\mathbb{R}$ and we will assume this from now on.

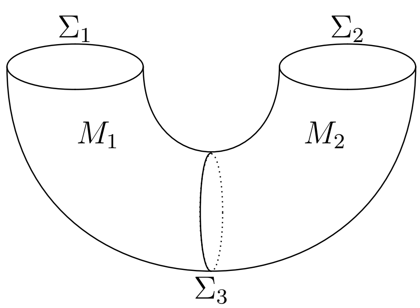
Now, the third axiom states that if $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup \Sigma_3^*$ and $M = M_1 \cup_{\Sigma_3} M_2$, as shown below, then we require the natural pairing

$$\langle \cdot, \cdot \rangle : Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2)$$

to be defined by

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle \tag{2.21}$$

where $Z(M_1) \in Z(\Sigma_1) \otimes Z(\Sigma_3)$ and $Z(M_2) \in Z(\Sigma_3)^* \otimes Z(\Sigma_2)$, hence, $Z(M) \in Z(\Sigma_1) \otimes Z(\Sigma_2)$.



Thus if $\Sigma_1 = \Sigma_2 = \emptyset$, i.e. the $(d+1)$-dimensional manifold M is closed, the pairing gives an element of Λ which is independent of the choice of Σ_3. This means that the numerical invariants of closed manifolds are independent of their decomposition $M = M_1 \cup_{\Sigma_3} M_2$ and can be computed in term of this decomposition via the above relation. Note that when $\Sigma_3 = \emptyset$, i.e. M is the disjoint union of M_1 and M_2, the pairing (2.21) reduces to

$$Z(M) = Z(M_1) \otimes Z(M_2)$$

This means that disjoint unions of $(d+1)$-manifolds are translated into tensor products of Λ-modules respecting the associations made for the distinguished elements $Z(M_i)$ $(i = 1, 2)$ to each component and extending it naturally to their tensor product. We have been working with this concept from the beginning on, associating a point in the plane to representations of $SU(2)$, in Section 1.1 or Hopf algebras in Section 2.2.3. In fact, the second and third axioms are used to view $Z(M_1)$ and $Z(M_2)$ as homomorphisms $Z(\Sigma_1) \to Z(\Sigma_3)$ and $Z(\Sigma_3) \to Z(\Sigma_2)$ respectively, [5], for instance, by $Z(M_1) \triangleright Z(\Sigma_1) = \Lambda \otimes Z(\Sigma_3) \cong Z(\Sigma_3)$ and $Z(M_2) \triangleright Z(\Sigma_3) \cong Z(\Sigma_2)$. Hence, (2.21) means that $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$ is transitive when cobordisms are composed. This corresponds to the previous assignment of the 3-valent vertex to a homomorphism between representations of (Hopf) algebras.

Now, consider $f \in Diff^+(\Sigma, \Sigma)$ and identify opposite ends of $\Sigma \times \mathbb{I}$ by f, such that we obtain a manifold Σ_f with

$$Z(\Sigma_f) = Tr(Z_f)$$

where $Z_f : Z(\Sigma) \to Z(\Sigma)$ is an induced automorphism. For example, we can construct $\Sigma \times S^1$ by identifying the opposite ends of $\Sigma \times \mathbb{I}$ and we obtain, [4],

$$Z(\Sigma \times S^1) = Tr(id_{Z(\Sigma \times \mathbb{I})}) = \dim Z(\Sigma \times \mathbb{I}).$$

Compare this result with the loop value in Section 1.1.

To finalize this section we describe shortly the physical interpretation of this theory. It is important to note, however, that there is no relation between the invariants $Z(M)$ and $Z(M^*)$ for closed $(d+1)$-manifolds given by the axioms. We can, however, consider the additional assumption that the vector spaces $Z(\Sigma)$ posses a non-degenerate Hermitian structure relative to some conjugation on Λ, which gives an isomorphism $Z(\Sigma^*) \to \overline{Z(\Sigma)}$, where $\overline{Z(\Sigma)}$ denotes $Z(\Sigma)$ with the conjugate action of Λ. This structure lets us consider a further Hermitian axiom,

$$Z(M^*) = \overline{Z(M)}$$

which means that $Z(M^*)$, regarded as a linear transformation between Hermitian vector spaces, is the adjoint of $Z(M)$. Hence, the numerical invariants of a closed manifold are sensible to changes of orientation, unless their value is real. Furthermore, with the Hermitian structure is possible to form a closed manifold $M \cup_\Sigma M^*$ from a manifold M with $\partial M = \Sigma$, such that

$$Z(M \cup_\Sigma M^*) = |Z(M)|^2$$

where the r.h.s. is the norm in the Hermitian metric, [4].

Physical interpretation of the axioms

In these axioms Σ is meant to indicate the physical space and the extra dimension in $\Sigma \times \mathbb{I}$ is the "imaginary" time. Then one can think of the space $Z(\Sigma)$ as the Hilbert space of the theory on Σ. The endomorphism $End(Z(\Sigma))$ given by $Z(\Sigma \times \mathbb{I})$ should be the imaginary time evolution operator e^{tH} where $t \in \mathbb{I}$, but the second non-triviality axiom does not allow any dynamics, since $Z(\Sigma \times \mathbb{I}) = id_{Z(\Sigma)}$ implies $H = 0$. There is, however, a "topological propagation" across a non-trivial cobordism M which changes the topology of Σ. Then, for a closed $(d+1)$-manifold M, the invariant $Z(M)$ is the partition function[28] given by some Feynman integral, i.e. with a special Lagrangian that gives rise to a topological invariant partition function. Relativistic invariance assures that the numerical invariants $Z(M)$ are independent of the decomposition of M, i.e. of the time variable chosen to slice the cobordism, [5].

The importance of this broad theory will be seen in the next chapter, where invariants of 3-manifolds and their calculation via $q-6j$-symbols are described.

[28]If $\partial M = \Sigma$, the distinguished vector $Z(M) \in Z(\Sigma)$ is interpreted as the vacuum state defined by the topology of M, [4].

Chapter 3

Invariants of 3-Manifolds

In this chapter we discuss the Turaev-Viro invariants of 3-manifolds and their relation to the concept of spherical categories. Most of the ideas are taken from [39, 7, 20].

In the first section, the invariant of a manifold is defined as a state sum based on quantum $6j$-symbols, which are associated with the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$. To define a state sum on a triangulation X of a compact 3-manifold M assume that there are colorings of X associating elements of the set of colors $\{0, 1/2, 1, ..., (r-2)/2\}$ with edges of the triangulation. This naturally leads to a one-to-one association of colored 3-simplexes of X with $q-6j$-symbols, which are multiplied[1] over all simplexes of the triangulation. The resulting weighted products are then summed over all colorings of X which are, in a sense defined below, admissible. These concepts lead to a 3-dimensional non-oriented topological quantum field theory where each closed surface F is associated with a finite-dimensional vector space $Z(F)$ over $\mathbb{C}$, as in the previous section. However, to define this vector space we have to fix a triangulation of F and show a posteriori that $Z(F)$ does not depend on the choice of triangulation, [39].

Although the state sums are computed on a triangulation of the manifold, they are independent of the choice of triangulation since some transformations of polyhedra, called Alexander moves, allow us to relate combinatorial equivalent triangulations leaving the evaluation of the state sum invariant. The number of transformations is infinite, however, in the case of triangulations of manifolds one can pass to the dual complex, called the cell subdivision. This dualisation, described in the second section, transforms the Alexander moves into certain operations on cell complexes, which can be presented as compositions of certain finite set of local moves. In a 3-manifold there are three such moves called the Matveev-Piergallini moves. This fact simplifies the task of checking the invariance of the state sums since there are only three identities to be verified. It turns out that these identities follow directly from the basic properties of the $q-6j$-symbols.

After redefining the state sum for the simple 2-polyhedra forming the cell subdivision, we give an informal identification of the constituent terms of the state sum with the diagrammatic language presented in Section 2.3. This identification allows us then to give an explicit expression of the invariant for the case when the objects used to construct it are representations of $U_q(\mathfrak{sl}_2)$ with q a root of unity.

Finally, a more general invariant of 3-manifolds is given briefly in Section 3.3, where

[1] More precisely, the "multiplication" is in fact a tensor contraction.

the only structure assumed is the one described in 2.2.3. Hence, the Turaev-Viro invariant defined in the next section is a special case which, in fact, satisfies two extra conditions. First, without going into detail, this invariant is defined for unoriented manifolds and second, that there exist a TQFT associated to the invariant given by the fact that each self-dual simple object of the category involved in the construction of the invariant is orthogonal[2], [7].

3.1 State Sum Invariants

First, the initial data and the conditions on it needed to define an invariant of 3-manifolds are given. Then, we proceed with the definition of the state sum models for closed 3-manifolds and its relation to topological quantum field theory is discussed.

For the initial data consider a commutative ring K with unity and denote by K^* the group of invertible elements of K. The data consists of five objects besides the ring K:

- A finite[3] set I of "colors".
- A function $f : I \to K^*$; $i \mapsto f(i) = w_i$.
- A distinguished element $w \in K^*$.
- A set adm of unordered triples of elements of I, $adm \subset I^3$, for which there are no further conditions imposed. The triples belonging to adm are called admissible.
- A set of ordered 6-tuples $(i, j, k, l, m, n) \in I^6$ which are admissible, meaning that the unordered triples (i, j, k), (k, l, m), (m, n, i), (j, l, n) are admissible. Furthermore, we assume that each of these 6-tuple is associated with an element of K called the symbol and denoted by
$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in K$$
These symbols are assumed to have the same symmetries as the usual $6j$-symbols in previous sections. From these symmetries we conclude that by permutation and interchange of the upper and lower arguments of any two columns respectively one can obtain different 6-tuples which correspond to symbols with the same value. Denote this common value of the symbols by $|T|$.

The initial data is assumed to follow four conditions. The first two of them axiomatise the orthogonality and the Biedenharn-Elliot identities for $q-6j$-symbols.

The data satisfies condition (I) if for any six elements $j_1, j_2, j_3, j_4, j_5, j_6$ in I such that (j_1, j_3, j_4), (j_1, j_3, j_6), (j_2, j_4, j_5) and (j_2, j_5, j_6) are admissible we have
$$\sum_j w_j^2 w_{j_4}^2 \begin{vmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{vmatrix} \begin{vmatrix} j_3 & j_1 & j_6 \\ j_2 & j_5 & j \end{vmatrix} = \delta_{j_4, j_6}. \tag{3.1}$$

[2]An self-dual simple object a is called orthogonal, if for its isomorphism $\phi : a \to \hat{a}$ we have $\phi = \hat{\phi}$.
[3]Notice here the importance of $U_q(\mathfrak{sl}_2)$ at a root of unity to make this color set invariant.

where we sum up over j such that the symbols involved in the sum are defined, i.e. the 6-tuples in the sum are admissible.

The data satisfies the condition (II) if for any pair of admissible 6-tuples $(j_{23}, a, e, j_1, f, b)$ and $(j_3, j_2, j_{23}, b, f, c)$ the following relation holds

$$\sum_j w_j^2 \begin{vmatrix} j_2 & a & j \\ j_1 & c & b \end{vmatrix} \begin{vmatrix} j_3 & j & e \\ j_1 & f & c \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ a & e & j \end{vmatrix} = \begin{vmatrix} j_{23} & a & e \\ j_1 & f & b \end{vmatrix} \begin{vmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{vmatrix}. \tag{3.2}$$

The condition (III) is satisfied if for any $j \in I$ we have

$$w^2 = w_j^{-2} \sum_{k,l:(j,k,l)\in adm} w_k^2 w_l^2. \tag{3.3}$$

Finally, the initial data is said to be irreducible, if for any j, k in I there exists a sequence $l_1, l_2, \ldots, l_n$ with $l_1 = j, l_n = k$ such that the triple (l_i, l_{i+1}, l_{i+2}) is admissible for any $i = 1, ..., n-2$. If we have irreducible initial data satisfying condition (I), then the r.h.s. of (3.3) is independent of $j \in I$, i.e. condition (III) is automatically satisfied.

Now, consider a tetrahedron with edges labelled by elements of the set I. Such a 3-simplex will be called a colored tetrahedron and is said to be admissible if for any of its 2-simplexes A the labels, or colors, of the three edges in A are in adm. From this we can understand geometrically the notion of an admissible 6-tuple. As mentioned before, the symbols corresponding to the admissible 6-tuples in the initial data are regarded from a geometrical point of view as colored tetrahedra. Thus, in this case, admissibility means the condition for the existence of a tetrahedron with positive volume, as in Section 1.3. We stress here the fact that each admissible colored tetrahedron corresponds to *a set* of admissible 6-tuples. There are 24 admissible 6-tuples for a given tetrahedron T, which may be obtained from each other by the obvious action of the symmetry group S_4 of T.

We now proceed to discuss the state model for **closed** 3-manifolds, which leads to an invariant of the manifold with respect to triangulations. Consider a closed triangulated 3-manifold M. Let M have b edges denoted by $E_1, E_2, \ldots, E_b$. A coloring of M is defined to be an arbitrary mapping $\phi : \{E_1, E_2, \ldots, E_b\} \to I$, i.e. we label all edges of a given triangulation of M. Denote the admissible colorings[4] of M by $adm(M)$. It is obvious that each $\phi \in adm(M)$ induces an admissible coloring of each tetrahedra T_i of M, denoted by T_i^ϕ.

As we saw in section 1.3.2 the $3n-j$-symbols can be expressed as a product of $6j$-symbols. The definition of a state $|M|_\phi$ of the manifold M is defined in the same fashion. For a given coloring $\phi \in adm(M)$ we set

$$|M|_\phi = w^{-2a} \prod_{r=1}^{b} w^2_{\phi(E_r)} \prod_{t=1}^{d} |T_t^\phi| \, ; \; |M|_\phi \in K \tag{3.4}$$

where a is the number of vertices and d the number of tetrahedra in M. Then, the invariant $|M|$ of M is defined as the sum over all admissible colorings for a given triangulation:

$$|M| = \sum_{\phi\in adm(M)} |M|_\phi. \tag{3.5}$$

[4]As before, admissible means that for any 2-simplex A of M the colors of its three edges form an admissible triple.

Next we present a theorem proved by Turaev and Viro, [39], giving a scheme to define topological invariants of 3-manifolds. In principle it is defined as (3.5), however, to realize the invariant one needs concrete initial data.

Theorem 33. *If the initial data satisfies the conditions (I), (II) and (III), then |M| does not depend on the choice of triangulation of M.*

The proof of this theorem can be found in [39, Sec. 5].

Now, let us consider the more general case where M is a **compact** triangulated 3-manifold. Suppose that e of the a vertices and the first f of the b edges of M lay on the boundary ∂M. All the same concepts as above apply here as well, so that a coloring of ∂M means an arbitrary mapping $\alpha : \{E_1, E_2, \ldots, E_f\} \to I$. The formula for a state $|M|_\phi$, however, has to be modified as follows to account for the boundary. For any $\phi \in adm(M)$ define

$$|M|_\phi = w^{-2a+e} \prod_{r=1}^{f} w_{\phi(E_r)} \prod_{s=f+1}^{b} w^2_{\phi(E_s)} \prod_{t=1}^{d} |T_t^\phi| \in K. \tag{3.6}$$

For $\alpha \in adm(\partial M)$, denote by $adm(\alpha, M) \subseteq adm(M)$ the set of all colorings $\phi \in adm(M)$ of M which extend α, i.e. which have α as a restriction of ϕ on ∂M. Define

$$\Omega_M(\alpha) = \sum_{\phi \in adm(\alpha, M)} |M|_\phi.$$

Hence, for an admissible coloring α of ∂M the invariant is $\Omega(\alpha)$ and it is dependent on the coloring α.

Theorem 34. *If the initial data satisfies the conditions (I), (II) and (III), then for any compact 3-manifold M with triangulated boundary and any admissible coloring α of ∂M, all extensions of the triangulation of ∂M to M yield the same $\Omega_M(\alpha)$, [39].*

This is a generalization of Theorem 33 and means that for a given triangulation and coloring of the boundary of a compact 3-manifold there is a state sum which is invariant under Alexander moves on the extensions of the triangulation of the boundary to all the manifold, i.e. on simplexes not lying on the boundary. In this case, the state sums are called relative invariants.

As mentioned above, this type of initial data relates to a topological quantum field theory (TQFT). In what follows we discuss these relations by describing the role that the invariants defined above take in the theory and how the modules associated with the boundary of a cobordism arise. The construction of a functor, which defines the TQFT is also discussed briefly following [39].

Consider a triangulated closed surface F. Since there is an element j of the set I attached to each edge of the triangulation and a function $f : I \to K^*$ with $j \mapsto w_j$ for each $j \in I$, each admissible coloring gives a set of elements of the group $K^* \subseteq K$ which corresponds by multiplication to an element of K. Hence, each triangulated closed surface F defines a K-module $C(F)$, which is the module freely generated over K by admissible colorings of F, i.e. each coloring gives an element of the set generating $C(F)$. If we equip $C(F)$ with a scalar product $C(F) \times C(F) \to K$ we can make the set of admissible colorings an orthonormal

basis of $C(F)$. According to the convention that there exists exactly one map $\emptyset \to I$, we set $C(F) = K$ if $F = \emptyset$, cf. Section 2.4.

Consider a cobordism $W = (M; F_+, F_-; i_+, i_-)$ between triangulated surfaces F_+ and F_- and define a homomorphism $\Phi_W : C(F_+) \to C(F_-)$ by

$$\Phi_W(\alpha) = \sum_{\beta \in adm(F_-)} \Omega_M(i_+(\alpha) \cup i_-(\beta))\beta \tag{3.7}$$

where $\alpha \in adm(F_+)$ and $i_+(\alpha) \cup i_-(\beta) \in adm(\partial M)$ is the coloring determined by α and β. From the above discussion regarding the construction of $C(F)$, we can regard Φ_W as a homomorphism having as matrix elements $\Omega_M(i_+(\alpha) \cup i_-(\beta))$ with respect to the natural bases of $C(F_\pm)$. For a closed[5] M, Φ_W acts in K as multiplication by $|M|$, while for a compact M, where the cobordism is $W = (M; id : \partial M \to \partial M, \emptyset \to \partial M)$, the $\Omega_M(\alpha)$ are the matrix elements for Φ_W. Hence, the invariants defined above give the homomorphisms defined in (3.7).

Thus, from theorem Theorem 34 one can conclude that for any cobordism $W = (M; i_+, i_-)$ between triangulated surfaces, the homomorphism Φ_W does not depend on the extension of triangulations of the surfaces to the triangulation of M.

Since each cobordism W between surfaces F_+ and F_- is a morphism $F_+ \to F_-$ of a category with objects closed manifolds, the composition of cobordisms $W_1 = (M_1; i_1 : F_1 \to \partial M, i_2 : F_2 \to \partial M)$ and $W_2 = (M_2; j_2 : F_2 \to \partial M, j_3 : F_3 \to \partial M)$ is again a cobordism $W_2 \circ W_1 = (M_1 \cup_{F_2} M_2; i_1, j_3)$ obtained by gluing M_1 and M_2 along F_2. From this, it is straightforward[6] to conclude that $\Phi_{W_2 \circ W_1} = \Phi_{W_2} \circ \Phi_{W_1}$ which describes the multiplicativity of invariants.

We have now the ingredients to construct the mentioned topological 3-dimensional QFT related to the initial data. For this, notice that the association $F \mapsto C(F), W \mapsto \Phi_W$ is not a functor since the induced homomorphism for the unit cobordism, which is simply the cylinder $F \times [0,1]$, is not always the identity. In [39] Turaev and Viro constructed a functor by building the quotient $Q(F) = C(F)/Ker\Phi_{id_F}$, where[7] $id_F = (F \times [0,1]; i_0, i_1)$. As a consequence of the multiplicativity of the invariants, $\Phi_W : C(F_+) \to C(F_-)$ induces a K-linear homomorphism $\Psi_W : Q(F_+) \to Q(F_-)$ which is also multiplicative and satisfies $\Psi_{id_F} = id_{Q(F)}$. Hence, $F \mapsto Q(F), W \mapsto \Psi_W$ is a functor from the category of cobordisms of triangulated 2-manifolds to the category of K-modules.

Since for any two triangulations of F there exists a triangulation of the cylinder $F \times [0,1]$ which coincides on $F \times 0$ and $F \times 1$ with these given triangulations, an isomorphism between the spaces $Q(F)$ defined via the triangulations of F is determined completely. Hence, $Q(F)$ does not depend on the triangulations up to isomorphism. In other words, each triangulation of F gives a K-module isomorphic to another module $Q'(F)$ given by a different triangulation of F. Moreover, the isomorphism does not depend on the triangulation of $F \times [0,1]$ either, so all modules $Q(F)$, for a given surface F, can be identified by this isomorphism. From this, we can conclude that the theory discussed here can be generalized even further to a functor

[5] Closed manifolds can be considered as cobordisms between empty manifolds.

[6] For a detailed discussion see [39].

[7] Here $i_t : F \to \partial(F \times [0,1])$ is defined by $i_t(x) = (x, t)$.

from the category of cobordisms of topological[8] surfaces to the category of K-modules. This theory is then called $(2+1)$-dimensional TQFT.

In the next section the Matveev-Piergallini moves are introduced as well as the diagrammatic correspondence up to a normalization factor to the invariants described in this section.

3.2 Moves on Triangulations, Simple 2-Polyhedra and TL-Recoupling Theory

To understand the independence of the constructions in the previous section we have to study some concepts on simplicial complexes and transformations of triangulations. We start with some basic definitions, and continue then with the dualisation of the triangulations of 3-manifolds and the Alexander moves to obtain simple 2-polyhedra and the Matveev-Piergallini moves. Finally, we give the diagrammatic form of the invariants, which is best understood in the dual version of the theory.

Definition 35. The **join** $X * Y$ of spaces X and Y is the quotient space of $X \times Y \times [0,1]$ obtained by contraction of subsets $pt \times Y \times 0$ and $X \times pt \times 1$.

This is a formal definition but a join $X * Y$ can be regarded as the union of segments joining X and Y such that any pair of segments intersect at most at their end points. The join of two simplexes is again a simplex, thus, triangulations of the spaces X and Y define a triangulation of their join, [40].

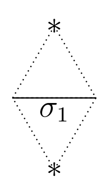
Definition 36. The **link** $lk(\sigma, K)$ of a simplex σ in a simplicial complex K is defined[9] by $lk(\sigma, K) = \{\tau \in K : \sigma * \tau \in K\}$, [23].

From the above definitions we are able now to define the concept of star of σ in K, which is the union of all closed simplexes containing σ and it is denoted $St(\sigma)$. More formal,

Definition 37. The **star** $St(\sigma)$ of a simplex σ is defined by $St(\sigma) = lk(\sigma) * \sigma$.

From this, the boundary of the star is $\partial St(\sigma) = lk(\sigma) * \partial\sigma$, [40].

To describe the transformations needed in this section, consider a link of a p-simplex σ_p in a triangulated n-dimensional manifold isomorphic to the boundary of a q-simplex, $lk(\sigma_p) \cong \partial\sigma_q$, where $n = p + q$. For example, in the case of a triangulated surface take a 1-simplex σ_1. In this case, the link is the set of the two vertices, denoted by $*$, opposite to σ_1 belonging to both 2-simplexes containing σ_1. Thus, the q-simplex mentioned above is, in this example, a 1-simplex containing these two points:



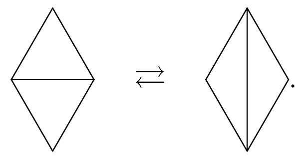
[8]The distinction here from the last sentence in the previous paragraph regarding the functoriality of Ψ is that the surfaces are non-triangulated.

[9]An equivalent definition would be that the link of a simplex σ is the union of all closed simplexes contained in its star which do not intersect σ, cf. [40].

Then, the boundary of the star of σ_p, $\partial St(\sigma_p) = lk(\sigma_p) * \partial\sigma_p$, is isomorphic to the join of the boundaries of a q-dimensional and a p-dimensional simplexes, i.e. $\partial St(\sigma_p) \cong \partial\sigma_q * \partial\sigma_p$. Now, the transformation, called simplex move of index p, is the replacement[10]

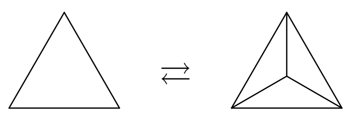
$$St(\sigma_p) \mapsto \sigma_q * \partial\sigma_p.$$

In our example, the transformation results in two 2-simplexes too, but with the q-simplex as the common edge:



The inverse transformation to the general one given above is the transformation of index $q = n - p$.

In the case where $p = n$ the transformation is a star subdivision centered at the given n-simplex σ_n, [40]. A star subdivision replaces the star $St(\sigma_n)$ of a simplex σ_n in a triangulation T by the cone[11] of $\partial St(\sigma_n)$ centered in a point $b \in \sigma_n$ leaving the rest of the triangulation $T \backslash St(\sigma_n)$ unchanged. For instance,



The above transformations are also called Alexander moves[12]. J. W. Alexander showed that for any dimensionally homogeneous[13] polyhedron P any of its triangulations can be transformed to any other by a finite sequence of Alexander moves, [39].

However, the number of Alexander moves is infinite[14] and, in the case of triangulations, one can not factorize them into a finite number of elementary ones. Thus, in order to verify the invariance of the above state sums under this type of transformations one has to dualise the moves in the sense described next. Each triangulation of a manifold M induces a cell subdivision dual to that triangulation, which can be constructed with help of the notion of barycenter of the simplexes involved as follows. Take a strictly increasing sequence $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_m$ of simplexes of the triangulation of a manifold M and associate an m-dimensional simplex $[\beta_0, \beta_1, \ldots, \beta_m] \subset M$ whose vertices are the barycenters of $\sigma_0, \ldots, \sigma_m$. For a simplex σ of M, the union σ^* of all simplexes $[\beta_0, \ldots, \beta_m]$ where β_0 is the barycenter of σ, is a combinatorial cell of dimension $\dim M - \dim\sigma$ called the barycentric star of σ. The cells $\{\sigma^*\}_\sigma$, where σ goes over all simplexes of M form a cell subdivision of M, cf. [39]. In a less technical way and in the here relevant 3-dimensional case, the dual cell complex to a tetrahedron is a collection of six 2-dimensional cells sharing a single vertex, which is the

[10]An extensive description of this type of moves can be also found in [23], where the transformations are called bistellar moves.

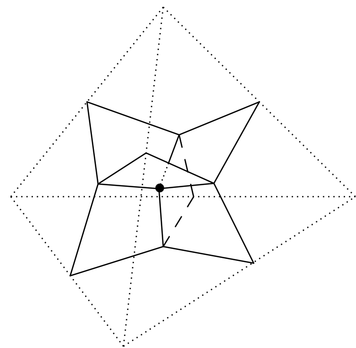
[11]A cone of K is defined to be the join of a simplicial complex K and a single point, [17].

[12]These moves are the dual version of the operations involved in the evaluation of spin networks in terms of $6j$-symbols via Moussouris' algorithm, cf. Section 4.2.1.

[13]This means that P is a union of closed simplexes of the same dimension.

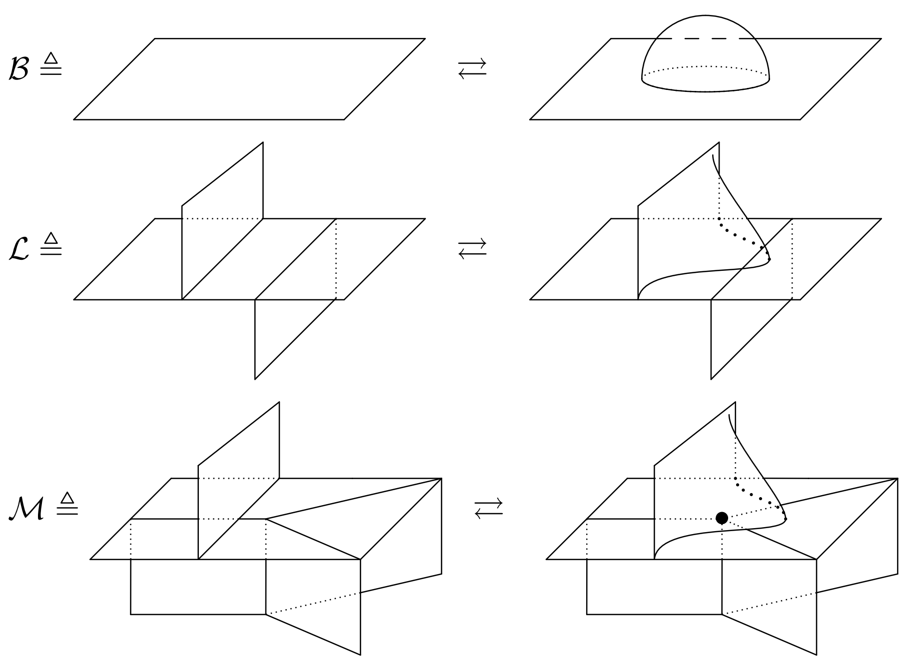
[14]They are determined by the combinatorics of the star of the simplex σ_p in a triangulated space, e.g. by the number of simplexes containing σ_p.

barycenter of the tetrahedron. Each of the edges of the tetrahedron intersects a 2-cell at exactly one point as in the figure below.



In this way, a triangulated 3-manifold gives rise globally to a dual cell complex with 3-cells homeomorphic to balls, called special spine, [20].

With the help of this dualisation it is possible to factorize the dual form of the star subdivisions. This is achieved by local modifications of the special spines called Matveev-Piergallini moves. There are three transformations of this kind, the bubble move[15] denoted by $\mathcal{B}$, the lune move denoted by $\mathcal{L}$ and the Matveev move denoted by $\mathcal{M}$:



The Matveev-Piergallini moves, however, do not act on the class of barycentric star subdivisions of triangulations so there is the need to enlarge the class of objects on which the state sums are defined. These objects, called simple 2-polyhedra, appear in a natural way

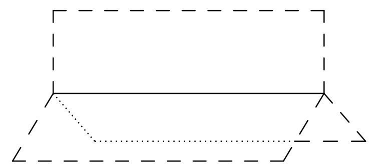
[15]In the case of three incident planes defining an edge, cf. (2) in def. 38, this move is called edge dilation. In the presence of $\mathcal{L}$- and $\mathcal{M}$-moves both versions of the $\mathcal{B}$-move are equivalent, [20, Ch. 10].

as 2-skeletons[16] of the cell subdivisions of compact 3-manifolds dual to triangulations, [39]. They also have a correspondence with objects of the recoupling theory defined in Section 2.3 which will be given after a short discussion about simple 2-polyhedra and the state sum defined for them. For an extensive discussion of the topic see the original papers by Turaev and Viro, [39] and [40].

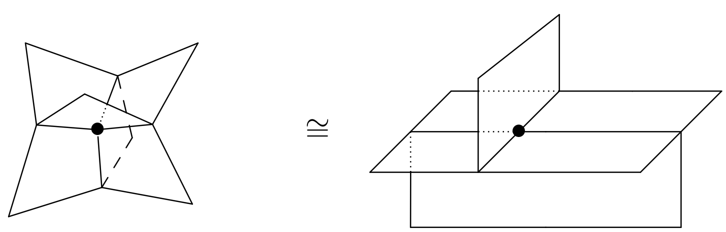
Definition 38. A 2-dimensional polyhedron (with boundary) X is **simple** if the neighborhood of each point of X is homeomorphic to either of the next spaces:

1. $\mathbb{R}^2$

2. The union of three half-planes meeting in their common boundary line

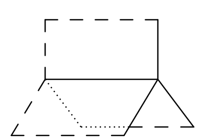


3. The cone over the one-skeleton of the tetrahedron[17]



4. The half-plane $\mathbb{R}^2_+$, or

5. the union of three quadrants $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ meeting in the half-line $x = 0$,



The points of X with neighborhoods homeomorphic to the last two ones above belong to the boundary ∂X, which is a simple graph[18].

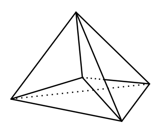
[16]Recall that given an inductive definition of a n-dimensional simplicial complex K, the m-skeleton of K is obtained by stopping at the m-th step.

[17]This is homeomorphic to six 2-dimensional cells sharing a single vertex, cf. [20, Ch. 10].

[18]A simple graph is a finite 1-dimensional CW-complex such that its 0-cells are trivalent vertices and its 1-cells are homeomorphic to $\mathbb{R}$, called edges, or to S^1 called loops.

The simple 2-dimensional polyhedra are naturally stratified. The k-strata ($k = 2, 1, 0$) being k-dimensional connected components of the set of internal points in X with neighborhood $\mathbb{R}^2$, (2.) and (3.) respectively. For the stratification of the boundary, the 1-strata are the edges and loops of the simple graph and the 0-strata are the 3-vertices of the simple graph.

Now we have the ingredients to redefine the previous state sum with the simple 2-polyhedra. The same concepts, like admissibility, apply but with the subtlety that the coloring is now with respect to the 2-strata of X[19]. Any coloring of X induces naturally a coloring of ∂X since a 1-stratum of the boundary acquires the color assigned to the 2-stratum of X in which this 1-stratum is contained. We denote the map defined by this construction by $\partial : adm(X) \to adm(\partial X)$. As mentioned before, the tetrahedra in Eq. (3.4) are associated to vertices, which carry a 6-tuple labelling the 2-strata that meet at the vertex. The association is such that if $x \in X \backslash \partial X$, the 1-skeleton of its corresponding tetrahedron T_x is the polyhedral link of x in X, i.e. the edges correspond to lines in the 2-strata and the vertices correspond to points in the 1-strata where the lines of the three 2-strata defining this 1-strata meet. This association labels automatically the edges of T_x. The figure below shows the six 2-strata intersecting in one vertex in the center of a tetrahedron made out of the germs of 2-strata defining the polyhedral link[20]:



The edges of the dual tetrahedron $\hat{T}_x$ correspond to edges of T_x, thus, they obtain the same labelling. This can also be seen, if one constructs the dual tetrahedron directly from the configuration of six 2-cells defining the vertex x, as in 38. From this construction it is clear that each admissible coloring of X induces an admissible coloring of $\hat{T}_x$. Then for $\phi \in adm(X)$ define

$$|X|_\phi = w^{-2\chi(X)+\chi(\partial X)} \prod_{s=1}^{f} w_{\partial\phi(E_s)}^{\chi(E_s)} \prod_{r=1}^{b} w_{\phi(\Gamma_r)}^{2\chi(\Gamma_r)} \prod_{t=1}^{d} |\hat{T}_{x_t}^{\phi}| \in K \tag{3.8}$$

where $\phi(\Gamma_i)$ is the color of the edge i of $\hat{T}_x$ corresponding to the 2-stratum Γ_i of X and $\partial\phi(E_s)$ is the color of the edge E_s of ∂X[21]. If E_s is homeomorphic to $\mathbb{R}$ then $\chi(E_s) = -1$ and if it is homeomorphic to S^1 then $\chi(E_s) = 0$. The corresponding invariant is

$$|X| = \sum_{\phi \in adm(X)} |X|_\phi,$$

and for any admissible coloring $\alpha \in adm(\partial X)$ we have

$$\Omega_X(\alpha) = \sum_{\phi : \partial\phi = \alpha} |X|_\phi.$$

[19]Admissibility is then w.r.t. the edge formed by three labeled 2-strata as in (2.) in the definition 38.

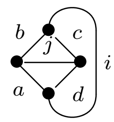
[20]Notice that the figure below is, in fact, the cone over the 1-skeleton of the tetrahedron. Hence, the polyhedral link is the 1-skeleton defining this cone.

[21]Recall that there are f edges E_s in the boundary of M, cf. Eq. (3.6).

From the conditions (I), (II) and (III) follows the invariance of $|X|$ and $\Omega_X(\alpha)$[22] under $\mathcal{L}$-, $\mathcal{M}$- and $\mathcal{B}$-moves respectively. A detailed proof of the invariance can be found in [39] and [20]. In the latter reference, the proof occurs in the framework of Temperley-Lieb recoupling theory in which the conditions (I) and (II) correspond to the orthogonality and Biedenharn-Elliott identity for $q-6j$-symbols respectively.

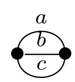
In the rest of this section we will discuss, following [20], the correspondence between the approach just discussed and another, which we will call Kauffman-Lins approach, involving the recoupling theory. The aim of this comparison is to understand the connections between these two frameworks. This is done[23] first by assigning weights to the vertices, edges and faces given a coloring of the special spine of a 3-manifold corresponding to X, cf. Eq. (2.17). Then, a partition function involving these weights is defined and finally the identification of the factors is made.

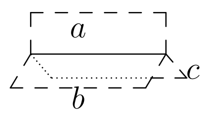
The first and most obvious association is, as explained before, that of a vertex x with a colored tetrahedron,

$|\hat{T}_{\sigma(x)}| \triangleq$ 

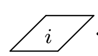
.

The weight of an edge in a special spine is the value $\theta(a,b,c)$ associated to it, where the labels are those of the three 2-cells incident to the edge, i.e. we have

$\theta(a,\, b,\, c) =$ 

$\triangleq$ 

As before, the 2-cells correspond in the dual sense to edges, hence, they carry only one color i. Therefore, the weight associated with it is the quantum integer Δ_i,

$\Delta_i \triangleq$ 

.

Definition 39. The partition function TV_{M^3} for a 3-manifold M in the Temperley-Lieb recoupling theory is defined by

$$TV_{M^3} = \sum_{\sigma} \prod_{v,e,f} \theta(\sigma(e))^{\chi(e)} \Delta_{\sigma(f)}^{\chi(f)} |\hat{T}_{\sigma(x)}|$$

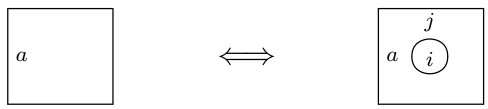
where $\chi(f)$ and $\chi(e)$ are the Euler characteristic of the 2-cell f and the edge e. If e has graphical nodes then $\chi(e) = -1$ and if e is a loop without nodes then $\chi(e) = 0$, cf. (3.8). The coloring σ involved in this definition of the partition function is over a finite color set $\{0, 1, \ldots, r-2\}$ as well, and admissible.

The behavior of TV_{M^3} under the bubble move deserves more attention since it helps us identify the rest of the factors in the definition of the Turaev-Viro invariant $|X|$. The result

[22]The moves $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{B}$ preserve the boundary, [39].

[23]The correspondence here is informal in nature. The formal proof is by S. Piunikhin, [32].

of performing a bubble move on a face of the special spine of M^3 turns out to be a global factor of $\Delta_a^{-1}\sum_{i,j}\Delta_i\Delta_j$, where a is the color of the face where the bubble move was realized and the sum is over all admissible triples (a,i,j). This factor can be explained as follows. Consider the bubble move on the face colored with a. Notice that the rest of the special spine is unaffected. We have then



where the colors i, j correspond to the surfaces inside the circle and to the hemisphere respectively, if we imagine the circle as being the intersection of the surface of an hemisphere j with the surfaces i and a. Since the cell colored by a obtains a hole in the process, its Euler characteristic is reduced by one. This explains the term Δ_a^{-1}. The factor involving the sum appears as a natural consequence of the summation over all possible admissible colorings keeping a fixed. There is no θ-value involved in this term since the edge defined by the 2-cells a, i, j is a loop without nodes.

There is in fact a relation between the weights Δ_j of the 2-cells for q-admissible triples (a,i,j) and (b,i,j) where $q = e^{i\pi/r}$ which strongly resembles condition (III) of the initial data. For $a,b,i,j \in \{0,1,\dots,r-2\}$, we have

$$\Delta_a^{-1}\sum_{i,j}\Delta_i\Delta_j = \Delta_b^{-1}\sum_{i,j}\Delta_i\Delta_j$$

where the sum is over the above q-admissible triples. Hence for any $a \in \{0,1,\dots,r-2\}$ and (a,i,j) q-admissible we have

$$\tau_q = \Delta_a^{-1}\sum_{i,j}\Delta_i\Delta_j = -\frac{2r}{(q-q^{-1})^2}.$$

The above relation looks just like the relation (3.3) for the elements of the commutative ring K. Thus we can identify $w^2 = \tau_q$ and for all $j \in \{0,1,\dots,r-2\}$ we have $w_j^2 = \Delta_j$, cf. [39].

Now, notice that the θ-values are in fact values of a vertex on the boundary of the 3-manifold with admissible coloring induced by the coloring of the special spine. These values correspond to products of terms like $w_{\partial\phi(E_s)}$ in the evaluation of $|X|_\phi$. This is readily seen if one recalls that the value of the theta-nets are combinatorial products of Δ_i in which admissible coloring is involved.

Finally, to obtain a topological invariant of M^3 from the recoupling theory one has to normalize the above defined state summation in order to take care of its change by the factor τ_q when considering bubble moves. Thus, the invariant of 3-manifolds in [20] is defined by

$$I_{M^3,q} = \tau_q^{-(t-1)}TV_{M^3}$$

where t is the number of 3-cells[24] in the decomposition of M^3. Thus, the factor $\tau_q^{-(t-1)}$ corresponds to the factor involving $w^2 \in K$ in $|X|$.

As mentioned before, the two invariants defined in this section coincide when q is a root of unity. The Kauffman-Lins approach gives the tools to calculate the invariant in a direct way while the Turaev-Viro approach gives us a broader insight about the theoretical structure giving rise to the invariant and its link to the topological QFT by the fact that a simple 2-polyhedron X is, in fact, a cobordism between simple graphs.

3.3 Invariants from Spherical Categories

In this section a more general version of the above discussed theory is presented following [7], where an algebraic framework for constructing invariants of closed oriented 3-manifolds is presented. We use the previously learned concepts in section 2.2.3 since the data for the construction of the invariant is a spherical category, for instance, the representations of the quantized enveloping algebra of $\mathfrak{sl}_2$ give the Turaev-Viro invariant defined in the previous section. In [7] the invariance from a finite set of moves on triangulations is also proved without going to the dual form of the transformations. Here we will only give a general account of the results obtained by J. W. Barrett and B. W. Westbury.

For the rest of this section by spherical category we mean an **additive** (strict) spherical category and we assume some conditions on it. First, the ring $\mathbb{F} \cong End(e)$, which is commutative in any additive monoidal category, is assumed to be a field. Second, each set of morphisms in our spherical category is a finite dimensional vector space (over $\mathbb{F}$).

In this framework the data for the state sum consists of the set of labels $I = J$, which is the set[25] of simple objects in the category, a set of state spaces and a set of partition functions for each tetrahedron. Denote by $D(a, b, c)$ the standard oriented triangle $+(012)$ labelled by $\partial_0 D \mapsto a, \partial_1 D \mapsto b, \partial_2 D \mapsto c$, where ∂_i is the map[26] sending any n-simplex σ in a simplicial complex to one of its $(n-1)$-faces obtained by omitting the i^{th} vertex of σ. The state space for $D(a, b, c)$ is a vector space $H(D_{a,b,c}) = H(a, b, c) = Hom(b, a \otimes c)$ over a field $\mathbb{F}$, whereas for the opposite oriented triangle $-D(a, b, c)$ it is defined to be the dual vector space $H^*(a, b, c)$, cf. Sec. 2.2.3. Consider once again the standard oriented tetrahedron $T = +(0123)$ with edges $\partial_i \partial_j T$ labelled by e_{ij}. The partition function corresponding to this tetrahedron is defined to be the linear map

$$\begin{Bmatrix} e_{01} & e_{02} & e_{12} \\ e_{23} & e_{13} & e_{03} \end{Bmatrix}_+ : H(e_{23}, e_{03}, e_{02}) \otimes H(e_{12}, e_{02}, e_{01}) \to H(e_{23}, e_{13}, e_{12}) \otimes H(e_{13}, e_{03}, e_{01})$$

and accordingly the partition function of the opposite oriented tetrahedron $-T$ is defined by

$$\begin{Bmatrix} e_{01} & e_{02} & e_{12} \\ e_{23} & e_{13} & e_{03} \end{Bmatrix}_- : H(e_{23}, e_{13}, e_{12}) \otimes H(e_{13}, e_{03}, e_{01}) \to H(e_{23}, e_{03}, e_{02}) \otimes H(e_{12}, e_{02}, e_{01}).$$

[24] Recall that 3-cells correspond to vertices in the dual form of the combinatorial manifold X. Thus, t is the number of vertices in X.

[25] More precisely, it is a set of representatives of each isomorphism class of simple objects.

[26] Note that these maps satisfy $\partial_i \partial_j \sigma = \partial_{j-1} \partial_i \sigma$ for $i < j$, cf. [7].

Hence, once again, we observe the correspondence between a topological object and a morphism, where the factors in the tensor products are associated to each one of the four faces of the tetrahedron[27].

This data determines an element $Z(M) \in \mathbb{F}$ for each labelled simplicial closed manifold M, the simplicial invariant of M obtained as follows. Let

$$V(M) \equiv \bigotimes_{D \in K(M)} H(D)$$

where $K(M)$ is the simplicial complex triangulating M, D are the triangles in this triangulation and $H(D)$ are the state spaces of the corresponding triangles. Consider the tensor product over the set of partition functions corresponding to each tetrahedron in $K(M)$. The resulting morphism[28] is a linear map $V(M) \to V^{\pi}(M)$, where the tensor product $V^{\pi}(M)$ is defined as $V(M)$ but with factors permuted by some permutation π. Furthermore, the iteration of the standard twist $P : x \otimes y \mapsto y \otimes x$ gives a unique linear map sending $V^{\pi}(M)$ back to $V(M)$. The composition of these two maps gives a linear map $V(M) \to V(M)$ and its trace defines the invariant $Z(M)$[29]. Then a state sum invariant of a closed manifold is then obtained by a weighted sum of these simplicial invariants over the class of (admissible) labelling.

As mentioned in section 2.2.3 the (additive) spherical category which defines the state sum model is the non-degenerate quotient category constructed from the category of representations of a spherical Hopf algebra. This construction is important, and always possible (see sec. 2.2.3), because in order to construct the invariants one needs to take a non-degenerate category since this property is needed for the semisimplicity condition which allows us to construct a well defined manifold invariant. The category of representations of the Hopf algebra may be degenerate but the quotient on page 48 is not. However, it is not the category of the representations of any finite dimensional Hopf algebra since it is not possible to assign a dimension to *each* object which would be additive and multiplicative under direct sum and tensor product respectively.

Given some isomorphisms of the label objects in a triangle $D_{a,b,c}$, i.e. $\phi_a : a \to a'$, $\phi_b : b \to b'$ and $\phi_c : c \to c'$, there is an induced isomorphism between the state spaces corresponding to $D_{a,b,c}$ and $D_{a',b',c'}$

$$Hom(b, a \otimes c) \to Hom(b', a' \otimes c')$$

given by $\alpha \mapsto \phi_b^{-1} \alpha(\phi_a \otimes \phi_c) = \alpha'$.

It follows that the map $V(M) \to V(M)$ is conjugated by the induced isomorphism on the state space of each triangle in the triangulation of the manifold M. Since the simplicial invariant mentioned above is the trace of this map and the trace is invariant under conjugation by a linear map we have, for a closed simplicial manifold M, that the invariant $Z(M)$ only depends on the isomorphism class of the labelling.

[27]Recall that the labelled trivalent vertex can be regarded in a certain sense as the dual of a labelled triangle.

[28]Recall the construction of an edge in Section 2.3 where the representations of the quantum group were woven into edges by permuting them in the order of their tensor product, cf. [20].

[29]Note that the notation given here and in Section 2.4 is no coincidence.

In fact, for a combinatorial[30] isomorphism of labelled manifolds $f: M \to N$ the simplicial invariants are equal. This follows from the pivotal structure of the spherical category which governs the properties of the state space of a triangle under combinatorial isomorphisms and the spherical property which allows isotopy on the sphere[31] and governs the properties of the non-degenerate pairing.

To end this chapter, we present the main result given in [7], which is stated for a more general type of 3-manifolds, however, we present it here as a result for closed 3-manifolds.

Theorem 40. *A finite semisimple spherical category of non-vanishing dimension determines an invariant of oriented closed 3-manifolds. In other words, any simplicial closed manifold M which triangulates a given piecewise-linear manifold $\mathcal{M}$ determines the manifold invariant given by*

$$C(M) = K^{-v} \sum_{l:E\to J} Z(M,l) \prod_{e\in E} dim_q(l(e))$$

where K is the dimension of the spherical category, v the number of vertices in M, the set of edges is denoted by E and J is a set of representatives from each isomorphism class of simple objects of the category. The map $l: E \to J$ is the labelling, thus, the labelled manifold is denoted by (M,l).

As before, this state sum invariant does not depends neither on the choice of simple objects J nor on the choice of the simplicial structure. The invariance under the choice of triangulation is a consequence of the invariance of $C(M)$ under bistellar moves which follows from the orthogonality and the Biedenharn-Elliot relation for the partition function corresponding to the tetrahedron.

To summarize, we started in Section 1.1 with the introduction of the classical model of Penrose's spin networks and its state sum as the Regge-Ponzano theory. The term "classical" here refers to the construction of the spin networks out of representations of Lie groups, $SU(2)$ in the case described in Section 1.1. However, the classical model has for some manifolds a divergent behavior coming from the infinite sum over the set of labels in the coloring of the triangulations. In order to avoid these divergences one has to construct the manifold invariant from the q-deformed universal enveloping algebra of the corresponding semisimple Lie algebra at a root of unity; for instance, the Turaev-Viro invariant can be seen as the regularization of the Regge-Ponzano state sum. Using this Hopf algebra has the effect that the set of representation labels is finite, hence, the state sum is finite, [15]. As mentioned before, this is a special case of the state sums given in Theorem 40, where the data comes from the more abstract notion of spherical categories.

[30] Combinatorial maps are maps of complexes, as opposed to simplicial maps which are combinatorial maps that preserve orderings. The difference is that a simplicial complex is a complex together with a total ordering of the vertices of each simplex. Thus, a single simplex has no symmetries, whereas the corresponding complex admits the permutations of its vertices as its symmetries. Therefore, in the above case, the combinatorial isomorphisms are representations of S_3.

[31] Any permutation of the vertices of a tetrahedron, i.e. elements of S_4, can be extended to an isotopy of the sphere.

Chapter 4

Non-planar Spin Networks

After giving an account of the physical motivation for spin networks, their relation to general relativity and placing them in an algebraical context as well as in the setting of TQFT we now turn to another language useful to describe some aspects of these objects. We will use some basic concepts of (topological) graph theory taken from [16, 9] to describe the embeddings of spin networks in surfaces, in particular the cellular embeddings of non-planar graphs with a $(3, 3)$-bipartite graph as subgraph.

Moussouris' algorithm for the evaluation of planar spin networks and the Decomposition Theorem are presented in the second section of this chapter. We will apply and extend these ideas to evaluate the above mentioned embeddings, which will allow us to define a toroidal symbol in order to attempt a generalization of the algorithm for the evaluation of non-planar networks. To achieve this a few identities for relating the evaluations of the different embeddings in the torus are given. We explain the main concepts involving the generalization of the evaluation of non-planar spin networks in terms of toroidal symbols, however, the process for obtaining the main result is still ongoing.

4.1 Kuratowski's Theorem and the Embedding of Graphs in Surfaces

A **graph** G is a pair of sets $(V_G; E_G)$ where $V_G \neq \emptyset$ is the set of vertices of G and E_G is a set of unordered pairs of elements of V_G which might be empty. The pairs of vertices are called edges and they are defined, in an abstract way, as a relation between the objects defining G. If two vertices form an edge, we say they are adjacent or neighbors. A **subgraph** of a graph G is a graph $H = (V_H; E_H)$ such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$, [16]. Another concept related to subgraphs are the **minors** of a graph; these are graphs obtained from G by a succession of edge-deletions and edge-contractions. If the minor M was obtained only by edge contractions, then G is said to be contractible to M, [9].

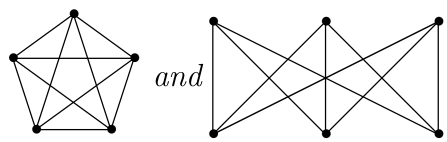
As mentioned before, we are able to consider only trivalent vertices since graphs of higher degree, i.e. with vertices of higher valence, are expandable to cubic graphs. On the other hand, a graph is only allowed to have at most one edge between two adjacent vertices. If two vertices are joined by more than one edge, the structure is called a multigraph. In the case of cubic graphs we only have trivalent vertices, hence, a multigraph would have two

vertices joined by at most three edges. This impose, however, no constraints in the class of spin networks since the double edge can be reduced to a single edge using 2.13 on page 55 and a triple edge is exactly the theta function defined previously as the value of a 3-vertex by 2.14 on page 56.

Two graphs are **homeomorphic** if they can be obtained from the same graph by subdividing its edges. Subdividing an edge $e = vw$ between two vertices v, w is the operation of inserting a new vertex z such that e is replaced by two new edges vz and zw, [9].

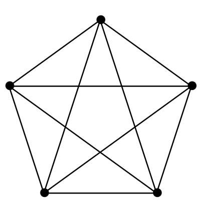
The drawing of a graph G in a (closed) surface S consists of points corresponding to vertices and simple curves, corresponding to edges, joining the points. If there are no crossings in the drawing, i.e. the curves do not meet except at their end-vertices, then the drawing is an embedding. The embedding is called **cellular** if each region is homeomorphic to an open disc. It is in this sense that planarity is defined, namely, a graph is planar if it can be embedded in a plane, hence, in the 2-sphere. Surprisingly, there is a simple criterion for determining whether a graph is planar or not, which is given by Kuratowski's theorem.

Theorem 41. ***Kuratowski's Theorem.*** *A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor, i.e. there are no subgraphs homeomorphic or contractible to K_5 and $K_{3,3}$.*

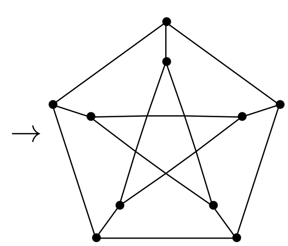


K_5 $K_{3,3}$

The graph K_5 is called the complete graph on five vertices and $K_{3,3}$ is the $(3,3)$-bipartite graph. Notice that the K_5 graph is 4-valent and can be expanded to obtain the Petersen graph which is a cubic graph with 10 vertices as depicted below. If one deletes any of its vertices and the edges incident to it, one finds a graph homeomorphic to the $(3,3)$-bipartite graph. Hence, we only need to focus on the latter graph.



K_5



Petersen graph

Closed surfaces, on the other hand, are categorized into orientable and non-orientable surfaces, e.g. the sphere S^2, torus T^2 or the real projective plane P^2. Any oriented surface

is homeomorphic to the sphere or to the connected sum $T^2 \# T^2 \# \ldots \# T^2$ of a finite number h of tori, while any non-orientable closed surface is homeomorphic to a connected sum of a finite number of copies of the real projective plane, where the empty sum is defined as P^2 itself.

Whether a given graph G is embeddable in an orientable surface S_h or not depends on its genus $\gamma(G)$, which is defined to be the minimum genus of any orientable surface in which G is embeddable, i.e. G is embeddable in S_h if $h \geq \gamma(G)$. In fact, any graph can be embedded in a surface with enough handles just by adding a handle at each crossing, but we are rather interested in cellular embeddings[1] for which Euler's formula hold,

$$v - e + f = 2 - 2h \tag{4.1}$$

where v, e, f are the number of vertices, edges and faces respectively and h is the (orientable) genus of S_h. In this context, a planar graph has genus $\gamma(G_{planar}) = 0$.

There is no general formula for calculating the orientable genus of a given graph, however, for the (s, r)-bipartite graph it is given by

$$\gamma(K_{s,r}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil \tag{4.2}$$

where $\lceil x \rceil$ denotes the next integer bigger than x. Hence, $K_{3,3}$ is embeddable in the torus but not in the sphere since $\gamma(K_{3,3}) = \lceil \frac{1}{4} \rceil = 1$, [9].

There are similar relations for the non-orientable case, however, our discussion will be only for embeddings in orientable surfaces since embeddings of the graphs in non-orientable surfaces involve a "twist" which is not clear how to deal with in the context of spin networks.

Notice that from the relation (4.1) there is a topological constraint to the allowed cellular embeddings for a given graph. For instance, $K_{3,3}$ has 6 vertices and 9 edges, thus, we obtain a constraint for the number of faces, $f = 5 - 2h$, since $f, h > 0$. Furthermore, from (4.2) we have $h \geq 1$, hence, $f = 3$ or $f = 1$ in the case where $K_{3,3}$ is embedded[2] in T^2 or $T^2 \# T^2$ respectively.

Is there other information encoded in the graph that can help us to further narrow down the possible embeddings? Does the orientation of the vertices impose a constraint on the embedding? Now, having found the number of possible faces (or 2-cells) for the embeddings, we want to construct oriented surfaces such that the cellular embeddings are automatically realized. In order to achieve this, we need to find the circuits of the given graph G and attach 2-cells to the regions bounded by them.

A **circuit** is a closed walk[3] in G such that no edge is repeated in the same direction. One may imagine a circuit as walking along an edge in a certain direction and, when getting

[1] In fact, if $\gamma(G) = h$, then every embedding of G on S_h is cellular, [9].

[2] From now on, whenever we refer to an embedding, it is meant a cellular embedding.

[3] A walk in G is an alternating sequence $v_1 e_1 v_2 e_2 \ldots v_{n-1} e_{n-1} v_n$ of vertices and edges of G, where every edge e_i is incident with v_i and v_{i+1}, and $v_i \neq v_{i+1}$. If $v_1 = v_n$, then it is a closed walk, [16]. It is important to notice here that this definition is a property of the graph itself. We will, however, abuse the use of the language and refer to the regions bounded by the circuits (cf. Theorem 42) as *embedded circuits*, when they have repeated edges in both directions, or as *embedded cycles* when no edges are repeated. To clarify, the difference is that the latter concepts are related to the embedding of the graph in a surface and the definition given above is a property of the graph related only indirectly to the embedding of the graph through Theorem 42.

to a vertex, the direction to follow (either clockwise or counter-clockwise) is given by the orientation, also called rotation, of the vertex under consideration. Thus, a given configuration of the orientations of all vertices in the graph, called a rotation scheme, induces a set of circuits giving rise to a specific cellular embedding. Hence, all embeddings can also be described by giving the orientation[4] of each vertex and specifying the regions bounded by the circuits obtained from applying the so called **rotation rule**: after the edge xy, take the edge yz, where z is the successor to x in the permutation of the neighbors of the vertex y, see examples below. The previous discussion is formalized in the next theorem, [9].

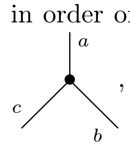
Theorem 42. ***Rotation Scheme Theorem.*** *Let G be a connected graph with v vertices and e edges, and let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_v\}$ be a set of cyclic permutations of the neighbors of the vertices $\{1, 2, \ldots, v\}$, i.e. a set of a given orientation of all vertices. Let $W_1, W_2, \ldots, W_f$ be the circuits obtained by applying the rotation rule to Π. Then the circuits are the boundaries of the regions of a cellular embedding of G in S_h, with $h = (2 - v + e - f)/2$, the genus of the orientable surface S_h. Hence, all possible embeddings of a graph are provided by the rotation schemes.*

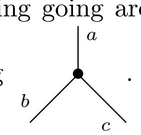
Let us consider the graph $K_{3,3}$ with $v = 6$, $e = 9$. This is a graph with trivalent vertices, hence, there are two possible rotations[5] for each of the six vertices. As a consequence, there are $2^6 = 64$ different sets $\Pi_{i=1,\ldots,64}$. The question that arises immediately is whether all these sets induce topological inequivalent embeddings or not. In other words, if we disregard the labeling of the vertices, how many different embeddings of $K_{3,3}$ in the torus or the double-torus exist?

Before making a general claim, let us work out some examples to illustrate the construction of embeddings by the rotation rule in order to understand the relation between the set of orientations and the embeddings induced by them. We will achieve this by listing the vertices and their neighbors and from this list extract the circuits in the embedding using the rotation rule, [9, 16].

Notice that for a general (r, s)-bipartite graph the defining characteristics are: *(i)* The r vertices corresponding to a set, say R, are adjacent to each of the s vertices corresponding to another set, say S, and *(ii)* $R \cap S = \emptyset$. In the case of $K_{3,3}$ each of the odd vertices, $R = \{1, 3, 5\}$, are adjacent to each of the even vertices, $S = \{2, 4, 6\}$. Thus, we denote (up to cyclic permutations) positive orientations of even and odd vertices as (135) and (246)

---

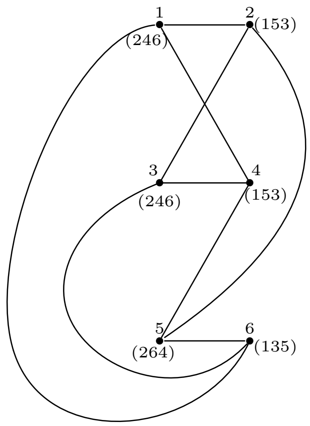
[4]The orientation is given as a cyclic permutation of the neighbors encountered while going clockwise around the vertices. In this convention, the orientation of a vertex v is the equivalent class of even permutations, in the case of positive orientation denoted $(+1)$, or odd permutations, in the case of negative orientation denoted (-1), of its neighbors. Such a definition takes into account the fact that both directions, anticlockwise and clockwise, can be described by listing the neighbors in order of their appearance when going *clockwise*, for instance, if (abc) describes the positive orientation of 

, then (acb) describes the negative orientation, meaning going around counter-clockwise, which can also be regarded as permutating the edges vc and vb giving 

. Notice that *in the diagram* the listing of neighbors of v is still clockwise.

[5]In fact, if a vertex v has degree d, then there are $(d-1)!$ different rotations of v.

respectively, and (153) and (264) for negative orientations. For instance, a standard way of picturing $K_{3,3}$ with minimal crossing is



This configuration has orientations (246) for the vertices 1 and 3, (264) for vertex 2 and (153) for the vertices 2 and 4, (135) for vertex 6.

Example 43. Consider the case where the vertices in each set have the same orientation, i.e. either $(+1, +1, +1)$ or $(-1, -1, -1)$. For instance, the vertices $\{1, 3, 5\}$ have negative orientation while the vertices $\{2, 4, 6\}$ have all positive orientation:

Vertex	1	2	3	4	5	6
Neighbors/Orientation	(264)	(135)	(264)	(135)	(264)	(135)

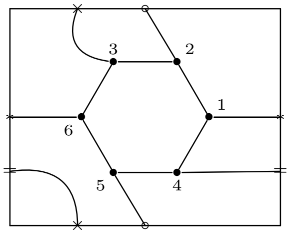
From this information we extract the circuits which will help us to construct the embedding corresponding to this configuration. For instance, take the edge (12) and apply the rotation rule on it, i.e. the neighbor of 2 coming after 1 in the cyclic permutation (135) is 3, hence, the next edge in the walk is (23). Apply again the rule to get (36) and so on. After some steps, depending on how long the walk is, one gets to the edge where the procedure started, meaning that one has to stop and apply the same procedure to another edge different than the ones encountered in the previous walk. In this specific configuration, this algorithm results in the following disjoint circuits,

1. $(12) \to (23) \to (36) \to (65) \to (54) \to (41) \to (12)$;

2. $(25) \to (56) \to (61) \to (14) \to (43) \to (32) \to (25)$;

3. $(21) \to (16) \to (63) \to (34) \to (45) \to (52) \to (21)$.

Notice that there is a difference between the "side" (xy) and (yx) reflecting the direction of the walk, hence, there are 18 "sides" available to build the circuits. In this case, there are three regions with six sides as boundaries, thus, the embedding has three faces and corresponds to an embedding in the torus as depicted below:



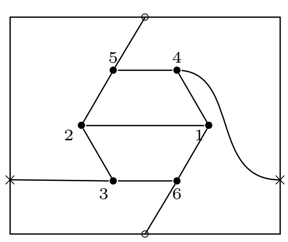
Example 44. Consider the case where one vertex has the opposite orientation relative to the two other vertices of the same set. For instance, the case where vertex 1 and 2 have positive orientation, (246) and (135) respectively, and the rest have negative orientation:

Vertex	1	2	3	4	5	6
Neighbors/Orientation	(246)	(135)	(264)	(153)	(264)	(153)

Using the rotation rule we obtain the following circuits,

1. $(12) \to (23) \to (36) \to (61) \to (12)$;
2. $(21) \to (14) \to (45) \to (52) \to (21)$;
3. $(32) \to (25) \to (56) \to (63) \to (34) \to (41) \to (16) \to (65) \to (54) \to (43) \to (32)$.

Notice that this time, we obtain two circuits of length four and a single one of length ten. Hence, the 18 sides available form three faces and the embedding is in a torus:



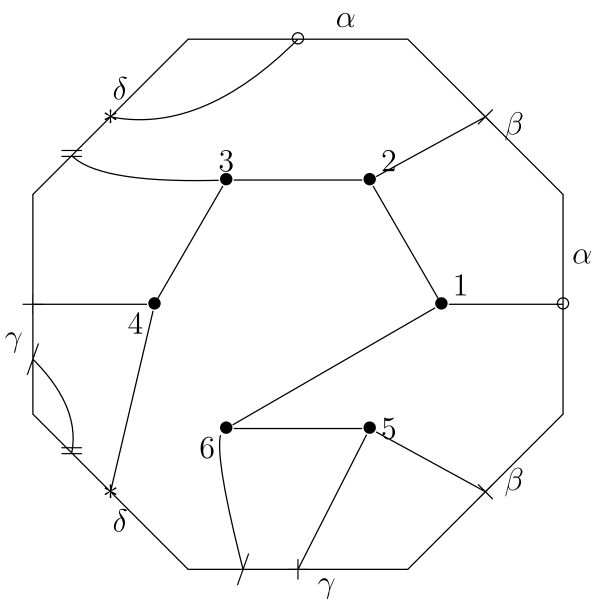
Example 45. Finally, consider the case where the orientation of all vertices in one set is the same while in the other set we have one vertex with the opposite orientation relative to the other two vertices. For instance, the case where all odd vertices have orientation (246) and vertex 2 has positive orientation as well, while the vertices 4 and 6 have orientation (153):

Vertex	1	2	3	4	5	6
Neighbors/Orientation	(246)	(135)	(246)	(153)	(246)	(153)

In this case we obtain, after using the described algorithm, only one circuit with 18 sides,

$$(12) \to (23) \to (34) \to (41) \to (16) \to (65) \to (52) \to (21) \to (14) \to (45) \to (56) \to \dots$$
$$\dots \to (63) \to (32) \to (25) \to (54) \to (43) \to (36) \to (61) \to (12)$$

where all sides are walked exactly once[6]. This configuration thus corresponds to an embedding in $T^2 \# T^2$ which can be represented in a plane in a similar manner as the torus,



where the Greek letters denote the borders of the frame that have to be glued together in order to obtain a double torus and the symbols on them identify corresponding points.

Now, define the **value $\mathfrak{v}$ of a set of vertices** as the modulus of the sum of orientations ± 1 of the vertices in that set, e.g. the value of $\{1, 3, 5\}$ with orientation $(-1 - 1 - 1)$ is $|-3| = 3$. The definition is such that, if the orientations of all vertices in a given set change, then the value remains invariant. For instance, one can achieve a change of orientation of all vertices in, say, the set $R = \{1, 3, 5\}$, e.g. $(+1+1-1)$, by an odd permutation of the vertices in $S = \{2, 4, 6\}$ such that $(+1+1-1) \mapsto (-1-1+1)$ but $\mathfrak{v}_{++-} = |+1| = |-1| = \mathfrak{v}_{--+}$. In fact, by permutations of the vertices in a set, one can construct all equivalent diagrams[7], i.e. giving the same embedding, since these operations do not change the 2-cells of the embedding, it merely results in a permutation of the vertices in it.

Observe that the three cases in the examples above are the only cases possible if we consider only the *relative* orientation between vertices of the same set, in which case the value is either $\mathfrak{v} = 3$, when all vertices have the same orientation, or $\mathfrak{v} = 1$, when one of the vertices has the opposite orientation relative to the other two in the same set. The value of each set is independent of each other, hence, we have the cases where the pair of values are $(3, 3)$, $(3, 1)$, $(1, 3)$ and $(1, 1)$. However, since $K_{3,3}$ is symmetric under exchange of sets $R \leftrightarrow S$ preserving the orientation, we can regard $(1, 3)$ and $(3, 1)$ as equivalent cases.

Another way of looking at this is to consider the partition of 18 in summands with some constraints. Each of the summands represents a circuit and their value represent the length of the circuit. The defining characteristics of the $(3, 3)$-bipartite graph do not allow the construction of circuits with an odd number of edges since this would mean that two vertices of the same set are adjacent. Thus, the partition of 18 cannot contain any odd

[6] Notice that a circuit induced in this way may have repeated vertices and edges used in both directions, however, if the edge is repeated in the same direction the algorithm must stop, [16].

[7] One has to consider the operation $R \rightleftarrows S$ as well.

numbers. It follows that the smallest possible circuit has length 4. Furthermore, the only number of summands in the partition can be 1 or 3 since they represent the regions of the embeddings. From these restrictions we conclude that the only partitions of 18 allowed are 18, 6+6+6, 4+4+10 and 4+8+6. However, the latter partition is not realized. To see this, notice that it is not possible to construct a 4-circuit which is not a 4-cycle[8], since this would mean that either one edge is repeated, in which case the edge would need to have two loops at each vertex, or two edges are repeated, this is not possible since all vertices in the graph are 3-valent. Thus, the only possibility is to have a 4-cycle, which implies automatically the existence of another 4-cycle. To see this, notice that the 4-cycle has two vertices of each set, therefore, there are two more vertices available to construct the graph, one of each kind. The defining characteristics of $K_{3,3}$ impose the constraint that these two vertices must be adjacent to each other and to the corresponding vertices in the original 4-cycle. This leaves no other possibility but to construct another 4-cycle, in contradiction to the partition $4+8+6$.

Remark. Observe the symmetry of the bipartite graph under permutation of its vertices reflected in the pair of values of the sets as well as in the partition of 18, thus, we may call the $(3,3)$ case "maximal symmetric", the $(1,1)$ case "minimal symmetric" and the $(3,1)=(1,3)$ case "asymmetric". Therefore, we can think of this pair of values as the "degree of symmetry" of the graph.

From the examples 43, 44 and 45 we see that the partitions $[6+6+6]$, $[4+4+10]$ and 18 correspond to the pair of values $(3,3)$, $(1,1)$ and $(1,3)$ respectively. We say that the maximal (minimal) symmetric graph has a $[6+6+6]$-type ($[4+4+10]$-type) embedding and the asymmetric graph has a 18-type embedding. Therefore we can say that the type of embedding is only dependent on the "degree of symmetry" of the graph given by the pair of values of the two sets R and S. In other words, the embeddings are topological invariant under permutations acting on the sets of even and odd vertices. Odd permutations on one set, merely change the orientation of all vertices in the other set, in which case the value of the set is not affected. Even permutations on one set only affect the cyclic order of the orientations in that set, e.g. if we have an orientation $(-1+1+1)$ of the set R an even permutation acting on R would only result in, say, $(-1+1+1)\mapsto(+1+1-1)$.

Thus, from all 64 possible configurations of the orientations of vertices in $K_{3,3}$ only three of them induce inequivalent embeddings. If the pair of values is $(3,3)$, then there are 4 equivalent configurations which induce a $[6+6+6]$-type embedding; either all 6 vertices have positive (or negative) orientation or 3 vertices from one set have positive (or negative) orientation while the vertices from the other set have opposite orientation. Therefore, we are left with $64-4=60$ configurations; 36 from them belong to the case where the pair of value is $(1,1)$. This is a $[4+4+10]$-type embedding, hence, one of the vertices on each set has the opposite orientation relative to the vertices from the set which belongs to, i.e. the sets have the orientations of the form $(+1-1-1)$ and cyclic, or of the form $(-1+1+1)$ and cyclic. Therefore, for each relative orientation there are 3 cases, which make $3\times 2=6$ for each set. The rest 24 of the configurations belong to the case where the pair of values is $(3,1)=(1,3)$. This gives an embedding in the double torus. There are 6 cases where the value of a set is 1 and 2 cases where the value is 3, hence, for each of the pairs $(1,3)$ and $(3,1)$ there are 12 configurations to consider.

[8] A **cycle** is a circuit which does not have any repeated edges in any direction.

Finally, we can summarize the above discussion by the following claim,

Claim 46. If the value of the two disjoint sets of vertices in $K_{3,3}$ is unequal, then the cellular embedding of $K_{3,3}$ corresponds to an embedding in $T^2 \# T^2$; otherwise the only cellular embeddings of $K_{3,3}$ are in the torus T^2, such that it is a $[6+6+6]$-type embedding for the $(3,3)$-value or a $[4+4+10]$-type embedding for the $(1,1)$-value.

This result is important since it would allow us to extract information of the terms needed in the evaluation of non-planar spin networks to account for their topology[9]. The graph contains topological information about the surface in which it is cellular embeddable and we can use recoupling theory to extract that information. The reason for this is that we are considering only cellular embeddings and we use all the information contained in the graph (number of edges, vertices and their orientation) to build the surfaces by the Rotation Scheme. Hence, the information of the topology of the surface must be contained in the graph itself; in other words, by reducing the graph in the embedding, we receive a factor in the evaluation that reflects the information of the graph being non-planar. That is why it is important to consider only cellular embeddings, the faces are only 2-cells homeomorphic to discs with no information about the global topology. For instance, in the case of the tetrahedron we have two cellular embeddings in the torus, one with an embedded 3- and another with an embedded 4-cycle as cells and both wrapping the two circles of the torus. Both cellular embeddings give, in fact, different evaluations, however only up to a constant involving powers of q (or A). In fact, these spin networks are contained in $K_{3,3}$, in the sense that reducing the graph of $K_{3,3}$ in the torus via Moussouris' algorithm leads to such diagrams. We will call the embedding of the tetrahedron in the torus with an embedded 3-cycle the *toroidal Racah coefficient*.

There are of course (non-cellular) embeddings of a graph in surfaces with higher genus, however, it is not the graph containing the information about the topology of the surface but the surface itself. If we consider a non-cellular embedding in the torus of the complete graph on 4 vertices, i.e. the tetrahedron, and we "cut" the surface along the edge of the graph wrapping the circle of the torus, we will get a surface which is not homeomorphic to a disc and which contains the information about the topology of the torus. Thus, the graph in that configuration has no information about a non-trivial topology.

Due to the classification of closed oriented surfaces we expect that the information extracted from the torus is sufficient to extend Moussouris algorithm for the evaluation of planar spin networks to the non-planar case. We believe that by knowing the evaluation of the spin network corresponding to the torus we can use it to evaluate all spin networks with higher genus in terms of products of this evaluation. To evaluate these spin networks it would be necessary to arrange them such that it is possible to "cut" their components (using the generalized Wigner-Eckart theorem, cf. [28]) corresponding to each handle of the oriented surface and evaluate each torus separately, this would give hopefully a sum of products of toroidal symbols.

[9]We mean by the topology of a graph, the topology of the surface in which the graph is embedded.

4.2 The Evaluation of Non-planar Spin Networks

In this section we discuss Moussouris' Decomposition Theorem, give an improved version of it and present its algorithm for the evaluation of planar spin networks, which relates these objects to the Ponzano-Regge partition function. We then apply this algorithm to the graph $K_{3,3}$ in order to extract the information needed to extend the algorithm to non-planar spin networks, i.e. we give the explicit form of the toroidal phase factor for the q-deformed case, and discuss what needs to be done to achieve such a generalization.

4.2.1 The Decomposition Theorem

In [28] J. P. Moussouris proved a theorem which relates the spin networks with the Ponzano-Regge theory by reducing a recoupling graph[10] to a sum of products of Racah coefficients. This reduction, known as the Decomposition Theorem, gives an evaluation of the spin network only dependent on the labelling of the graph, as in Chapter 3 for a manifold with boundary.

There are two versions of the mentioned theorem which we will present and analyze in this section in order to understand how the expansion of the algorithm for evaluating non-planar spin networks could be done. The first version of the theorem, called network version, is more general than the second version since it does not assume the spin network to be planar, however, it assumes implicitly the existence of an embedded cycle for the recoupling graph F to be reduced. This implies that the embedding of the graph in some surface has at least two 2-cells since a cycle induces a region homeomorphic to a disc by using only one side of each edge in the Rotation Scheme.

Moussouris does mention the importance of the orientation of the vertices in the evaluation of the graph, pointing out that considering the orientation of the vertices results in so-called phase factors, which can be isolated as values of graphs with two vertices with the same orientation, [28]. However, in the proof of the first version of the theorem this consideration enters only in the first and second steps of the induction on the number of vertices V in F, disregarding the fact that the orientation of the vertices of a recoupling graph affects the embedding of it in a surface, which might be such that there is no embedded cycle at all. This would mean that the spin network cannot be reduced straightforward. We will discuss this case later. Moreover, in the proof it is also assumed implicitly that after reducing all embedded cycles the only diagram left is either a Racah coefficient or a toroidal phase factor (cf. Section 4.2.2); in the latter case we can call such a recoupling graph a toroidal spin network since the phase factor left at the end of the reduction contains the information of the graph being embedded in the torus[11].

As seen in example 45, the appearance of an embedded cycle is not always the case and there exist spin networks which are irreducible if we only consider the operations described in [28], thus, Moussouris' Decomposition Theorem is limited to planar and toroidal spin networks. Hence, it is necessary to rewrite the Decomposition Theorem in a more precise

[10] Recall that a recoupling graph of a group G is a labelled 3-valent graph representing a contraction of tensors of G, cf. Sec. 2.3.2. In the following, the term "recoupling graph" will denote such a graph together with an orientation of its vertices.

[11] The Racah coefficient with a toroidal phase factor has as its cellular embedding exactly the one discussed at the end of the previous section.

manner in order to account for the case where the spin network is toroidal, i.e. for the phase factors, which are especially important for the q-deformed case.

We will now give both versions of the theorem and the proof of the network version following [28]. We modified the first version of the theorem to account for the discussion above. The second version is the special case where F is planar and it is proven by applying Schur's lemma and the Alexander moves to the graph-theoretic dual version of F, which gives a triangulation of the sphere. This dual version allows the connection with the Regge-Ponzano theory.

Theorem 47. Decomposition Theorem:

A recoupling graph F of a compact semisimple group G, which is at most toroidal, can always be evaluated as a sum of products of Racah coefficients of G and a toroidal phase factor.

Proof. The proof is by induction on the number of vertices V in F and the size l of the smallest cycle.

If $V = 2$, the recoupling graph is a toroidal phase factor or a theta-evaluation of a vertex. The case $V = 3$ is not possible.

If $V = 4$, the recoupling graph is a (toroidal) Racah coefficient or two phases.

If $V > 4$, we look for the smallest cycle in F and reduce it as follows, depending on the size l of it. A 2-cycle is reduced using Schur's identity 2.13 on page 55. This results in a new graph containing $V - 2$ vertices. A 3-cycle is eliminated by producing a single Racah coefficient by the Wigner-Eckart theorem. Alternatively, one can regard the so-called "crossing identity" described below to reduce the 3-cycle to a 2-cycle and apply Schur's identity. The resulting graph contains $V - 2$ vertices.

For the case $l > 3$ we have a cycle with l edges labelled by $j_1, j_2, \ldots, j_l$. This reduces to a $(l-1)$-cycle by the crossing identity derived from using 2.16 on page 58 on the edge, say, j_l. This operation results in a Racah coefficient multiplied by a recoupling graph in which the edge j_l is removed while a new "internal" edge x is introduced, coupling j_1 to j_{l-1} and the other two edges, which were coupling to j_l, are also coupled to x and to each other. This resulting product is summed over the new edge x as in the Recoupling Theorem 2.16 on page 58. The cycle is then reduced until $l = 3$.

This process of vertex reduction is repeated until $V = 4$ giving as a result a product of Racah coefficients and a phase factor summed over all internal variables. □

For completeness we give the second version of the theorem above,

Theorem 48. Planar version of the Decomposition Theorem:

*Let F be a **planar** recoupling graph and $D(F)$ its dual relative to a particular embedding in the sphere. Let $C(F)$ be a combinatorial 3-manifold produced by dissecting $D(F)$ with internal edges $x_1, x_2, \ldots, x_p$ into tetrahedra $T_1, \ldots, T_q$. Then, the evaluation of the recoupling graph is given by the amplitude*

$$\Psi(F) = \sum_{x_1,\ldots,x_p} \prod_{j=1}^{p} [x_j] \prod_{k=1}^{q} [T_k]$$

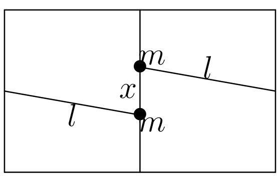
where $[x_j]$ is the loop-value of the edge x_j and the $[T_k]$'s are the Racah coefficient associated with the tetrahedra T_k.

The successive application of the Alexander moves, which correspond in the dual form to the elimination of a 3-cycle and the crossing identity, results in the introduction of sufficient internal edges to dissect the interior of $D(F)$ into tetrahedra, giving a combinatorial 3-manifold $C(F)$ with $D(F)$ as its boundary. This decomposition process is non-unique, however, the Biedenharn-Elliott identity and the orthogonality of the $6j$-symbols ensures the equivalence of the decompositions, [28]. This is a special case of the procedure to obtain the invariant[12] described in Chapter 3.

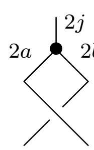
4.2.2 The Evaluation of the Toroidal Spin Network $K_{3,3}$

We will now apply the algorithm described in the above proof to the [4+4+10]- and [6+6+6]-type embeddings of $K_{3,3}$ on a torus, denoted by $K_{3,3}^{(1,1)}$ and $K_{3,3}^{(3,3)}$ respectively. This will be done in order to extract information for the evaluation of non-planar spin networks.

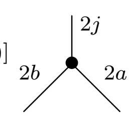
In the case of the embedding $K_{3,3}^{(1,1)}$ we may start by applying the crossing identity to the common edge of the 4-cycles and then eliminating the two resulting 3-cycles by extracting two Racah coefficients. The result is a sum over a single internal edge x of a product of three $6j$-symbols weighted by a factor of $(-1)^{2x}[2x+1]$. These are, however, not all the factors since the diagram left encodes the information of the graph being embedded in a torus. This diagram, which we will call **toroidal phase factor**, can be represented in a torus as follows:



If we "project" this diagram to the plane by connecting the loose ends of the edges m and l, once we have disregarded the frame of the above diagram, we get a theta-net with these edges crossing. In order to get a more familiar theta-net, which can then be set to have the value of 1, we need to "twist" the edge x. This is done by following operation on a vertex[13], [13, 20],



$$= (-1)^{a+b-j} A^{2[a(a+1)+b(b+1)-j(j+1)]}$$



$$. \qquad (4.3)$$

The result of applying Moussouris algorithm and the above twisting rule is a sum of products of Racah coefficients as in the planar case, however, the non-planar nature of the graph is reflected in the "twist factor" given above, i.e. in the evaluation of the toroidal phase factor. Thus, we have

$$\left[K_{3,3}^{(1,1)}\right] = \sum_x \Delta_x \left\{ \begin{array}{ccc} j_3 & j_4 & k \\ j_6 & j_1 & x \end{array} \right\} \left\{ \begin{array}{ccc} m & j_5 & j_6 \\ j_4 & x & l \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & l & j_1 \\ x & j_3 & m \end{array} \right\} (-1)^{l+m-x} A^{2[l(l+1)+m(m+1)-x(x+1)]} \qquad (4.4)$$

[12]Notice that in the amplitude given above the theta-net factors are missing. This is due to the fact that in [28] the spin networks are normalized such that the theta-nets are evaluated to one.

[13]Notice that here we are presenting the case of a vertex with an over-crossing, however, this operation is defined for an undercrossing as well. In this case we exchange $A \to A^{-1}$, cf. Section 2.3.

where $\Delta_x = (-1)^{2x}[2x+1]$ is the loop-evaluation 2.12 on page 55 and A is the square root of the deformation parameter q introduced before, cf. Sec. 2.1.3. The squared brackets denote the evaluation of a graph in terms of Racah coefficients.

The relation (4.4) looks similar to the $9j$-symbol. However, considering the cases where $A = \pm 1$, we have an overall factor of $(-1)^{l+m+x}$ which corresponds to one of the Racah coefficients having a vertex with the "wrong" orientation. This can be seen by expressing one of the $6j$-symbols where the labels x, l, m form an admissible triple in terms of $3j$-symbols and permuting the order of the labels in the corresponding $3j$-symbol by the following relation, [14],

$$(-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}.$$

The resulting factor is exactly the one described in [28, p. 65], i.e. a tetrahedron with one of the vertices having an orientation so that two edges cross. In fact, the diagram left after applying the crossing identity and eliminating only one of the two 3-cycle gives such a tetrahedron.

Consider now the evaluation $\left[K_{3,3}^{(3,3)}\right]$. It is possible to reduce the [6+6+6]-type embedding to the $[4+4+10]$-type one by applying the algorithm on two edges of the hexagonal figure shown in example 43 which belong to the same "exterior" 6-cycle, for instance the edges (12) and (54). From this procedure we get

$$\left[K_{3,3}^{(3,3)}\right] = \sum_{v,w} \Delta_v \Delta_w \begin{Bmatrix} j_4 & j_5 & l \\ m & v & j_6 \end{Bmatrix} \begin{Bmatrix} j_6 & j_1 & k \\ l & w & j_2 \end{Bmatrix} \left[K_{3,3}^{(1,1)}\right](v,w) \tag{4.5}$$

where the first $6j$-symbol is the result of the crossing identity on the edge $j_5 = (54)$ and the second one is the result of the same identity on the edge $j_1 = (12)$, and the factor $\left[K_{3,3}^{(1,1)}\right](v,w)$ corresponds to the relation (4.4) with $j_1 \to m$, $j_2 \to l$, $j_3 \leftrightarrow k$, $j_5 \to j_6$, $j_6 \to j_2$, $m \to w$, $l \to v$.

If we compare $\left[K_{3,3}^{(1,1)}\right]$ with the $9j$-symbol we may recognize the possibility to use the following relation between a $9j$-symbol (without twist factor) and $6j$-symbols, as in [14],

$$\sum_{\mu} (2\mu+1) \begin{Bmatrix} j_{11} & j_{12} & \mu \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} \begin{Bmatrix} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & \lambda \end{Bmatrix} = (-1)^{2\lambda} \begin{Bmatrix} j_{21} & j_{22} & j_{23} \\ j_{12} & \lambda & j_{32} \end{Bmatrix} \begin{Bmatrix} j_{31} & j_{32} & j_{33} \\ \lambda & j_{11} & j_{21} \end{Bmatrix} \tag{4.6}$$

However, this relation does not account for the twist factor, hence, it is not possible to use straightforward.

Claim 49. If we define the *toroidal symbol*

$$\begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix}_A := \left[K_{3,3}^{(1,1)}\right] = \sum_x \Delta_x \begin{Bmatrix} j_{11} & j_{21} & j_{31} \\ j_{32} & j_{33} & x \end{Bmatrix} \begin{Bmatrix} j_{12} & j_{22} & j_{32} \\ j_{21} & x & j_{23} \end{Bmatrix} \begin{Bmatrix} j_{13} & j_{23} & j_{33} \\ x & j_{11} & j_{12} \end{Bmatrix}$$
$$\cdots \times (-1)^{j_{21}+j_{32}-x} A^{2[j_{21}(j_{21}+1)+j_{32}(j_{32}+1)-x(x+1)]}.$$

Then the identity corresponding to (4.6) is

$$\sum_{\mu}[2\mu{+}1]\begin{bmatrix} j_{11} & j_{12} & \mu \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix}_A \left\{\begin{matrix} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & \lambda \end{matrix}\right\} = (-1)^{2\lambda}(-1)^{j_{21}+j_{32}-\lambda}A^{2[j_{21}(j_{21}+1)+j_{32}(j_{32}+1)-\lambda(\lambda+1)]}$$

$$\cdots \times \left\{\begin{matrix} j_{21} & j_{22} & j_{23} \\ j_{12} & \lambda & j_{32} \end{matrix}\right\}\left\{\begin{matrix} j_{31} & j_{32} & j_{33} \\ \lambda & j_{11} & j_{21} \end{matrix}\right\} \quad (4.7)$$

where $[2\mu+1]$ corresponds to the quantum integer defined in section 2.1.3 and equation 2.12 on page 55.

Proof. The only term containing μ in the expansion of the l.h.s. in term of $6j$-symbols is of the form

$$\sum_{\mu}[2x+1][2\mu+1]\left\{\begin{matrix} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & x \end{matrix}\right\}\left\{\begin{matrix} j_{11} & j_{12} & \mu \\ j_{23} & j_{33} & \lambda \end{matrix}\right\} = \delta_{x,\lambda}.$$

Thus, the only term left is the one on the r.h.s. □

Remark 50. Notice that (4.7) only holds if the toroidal symbol has that exact form, i.e. μ must be in any counter-diagonal position. Labels in that position appear in the expansion (4.4) only in one $6j$-symbol, thus, the orthogonality of the $6j$-symbols may be used straightforward. Moreover, it is possible to transpose the toroidal symbol since this only changes the ordering of the admissible triples in the $6j$-symbols, i.e. it is possible to use the symmetry properties of the $6j$-symbols to achieve a transposition of the toroidal symbol. In the regular case without twist factor, the $9j$-symbols have some symmetries and this constraint does not appear. However, the symmetries of the above defined symbol, if any besides the transposition, are not clear at the moment.

Even without having the symmetries needed it is possible to transform $\left[K_{3,3}^{(1,1)}\right](v,w)$ in order to bring it in a form suitable for the use of (4.7) to achieve a further simplification of $\left[K_{3,3}^{(3,3)}\right]$. Consider the following relation similar to the one given in [13][14],

$$\left[K_{3,3}^{(1,1)}\right](v,w) = \begin{bmatrix} j_4 & v & j_6 \\ k & l & w \\ j_3 & m & j_2 \end{bmatrix}_A = (-1)^{l+j_6-k-j_2}A^{2[l(l+1)+j_6(j_6+1)-k(k+1)-j_2(j_2+1)]}\begin{bmatrix} m & l & v \\ j_3 & k & j_4 \\ j_2 & w & j_6 \end{bmatrix}_A$$

Thus, using $1=(-1)^{-2(l+m+v)}$ for all v and the relation

$$\sum_{v}[2v+1]\left\{\begin{matrix} j_4 & j_5 & l \\ m & v & j_6 \end{matrix}\right\}\begin{bmatrix} m & l & v \\ j_3 & k & j_4 \\ j_2 & w & j_6 \end{bmatrix}_A = (-1)^{j_3+w+j_5}A^{2[j_3(j_3+1)+w(w+1)-j_5(j_5+1)]}\times\cdots$$

$$\cdots\times\left\{\begin{matrix} j_3 & k & j_4 \\ l & j_5 & w \end{matrix}\right\}\left\{\begin{matrix} j_2 & w & j_6 \\ j_5 & m & j_3 \end{matrix}\right\}$$

[14]The relation is given in the reference in a different form, namely, as a sum of products of three $6j$-symbols and a factor similar to the twist factor described above. We used the proof for the case $A=\pm1$ given in [13] as a guide to reconstruct the relation in order to present it as a "symmetry" of the toroidal symbol. Notice that by using this relation six times, one obtains the original form of the symbol, thus, this transformation can be regarded as a symmetry.

we can simplify (4.5) further.

Summarizing the discussion above we obtain the result that the [6+6+6]-type embedding of the $(3,3)$-bipartite graph is a spin network with following evaluation

$$\left[K_{3,3}^{(3,3)}\right] = \mathcal{A}(l, j_6, k, j_2; j_3) \sum_w \Delta_w \left\{ \begin{array}{ccc} k & j_6 & j_1 \\ j_2 & l & w \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & m & j_2 \\ j_6 & w & j_5 \end{array} \right\} \left\{ \begin{array}{ccc} j_4 & j_5 & l \\ w & k & j_3 \end{array} \right\} \times \dots$$
$$\dots \times (-1)^{w-j_3-j_5} A^{2[w(w+1)-j_3(j_3+1)-j_5(j_5+1)]}, \quad (4.8)$$

where $\mathcal{A}(l, j_6, k, j_2; j_3) = (-1)^{2j_3-(l+j_6+k+j_2)} A^{4j_3(j_3+1)} A^{2[l(l+1)+j_6(j_6+1)-k(k+1)-j_2(j_2+1)]}$.

Notice that the above relation is not exactly the toroidal symbol defined in (4.4), however, it looks very similar and it could be argued that it is, in fact, a toroidal symbol with an under-crossing instead of an over-crossing, cf. Footnote 13. It is yet unclear why the result is different and further work on this would need to be done. The difference could be related to the fact that there is a certain arbitrariness when it comes to project the diagram embedded in the torus into the plane, hence, the need to "choose" which (and even how) edges will cross. This corresponds to choose the orientation of the surface in which the spin network is embedded. However, the general form of the symbol remains and we can observe that the topology of the surface in which the diagram is embedded is refelected in the evaluation of this spin network.

Recall that we reduced the graph by choosing two common edges of the same embedded 6-cycles. If we reduce the diagram by applying the crossing identity to two edges of the central hexagonal region in example 43 belonging to two different 6-cycles in the exterior of this hexagon, e.g. $(14), (45)$, then the resulting reduction is given by a relation of the following form,

$$\left[\tilde{K}_{3,3}^{(3,3)}\right] = \sum\nolimits_{x,y,z} \Delta_x \Delta_y \Delta_z \left\{ \begin{array}{ccc} j_5 & j_6 & m \\ k & x & j_1 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & z & j_3 \\ m & j_2 & l \end{array} \right\} \left\{ \begin{array}{ccc} j_4 & j_5 & l \\ x & y & j_1 \end{array} \right\} \left\{ \begin{array}{ccc} y & j_4 & j_1 \\ j_3 & z & k \end{array} \right\} \times \dots$$
$$\dots \sum\nolimits_w \Delta_w (-1)^{x+z-w} A^{2[x(x+1)+z(z+1)-w(w+1)]} \left\{ \begin{array}{ccc} m & y & w \\ x & z & l \end{array} \right\} \left\{ \begin{array}{ccc} m & x & k \\ z & y & w \end{array} \right\}$$

which can be simplified using, [13],

$$\sum_w \Delta_w (-1)^{x+z-w} A^{2[x(x+1)+z(z+1)-w(w+1)]} \left\{ \begin{array}{ccc} m & y & w \\ x & z & l \end{array} \right\} \left\{ \begin{array}{ccc} m & x & k \\ z & y & w \end{array} \right\} = \dots$$
$$\dots = (-1)^{k+l-y-m} A^{2[k(k+1)+l(l+1)-y(y+1)-m(m+1)]} \left\{ \begin{array}{ccc} m & x & k \\ y & z & l \end{array} \right\} \quad (4.9)$$

twice, first summing over w to get an expression that can be reduced further by the Biedenharn-Elliott identity on the internal edge x and the second time summing over y to obtain the following simplified expression

$$\left[\tilde{K}_{3,3}^{(3,3)}\right] = \tilde{\mathcal{A}} \sum_z \Delta_z \left\{ \begin{array}{ccc} j_4 & l & j_5 \\ m & j_6 & z \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & j_2 & m \\ l & z & j_1 \end{array} \right\} \left\{ \begin{array}{ccc} k & j_1 & j_6 \\ z & j_4 & j_3 \end{array} \right\} (-1)^{j_1+j_3-z} A^{2[j_1(j_1+1)+j_3(j_3+1)-z(z+1)]}$$

where $\tilde{\mathcal{A}}$ is a constant dependent on A similar to the one in (4.8). Hence, we have

$$\left[\tilde{K}_{3,3}^{(3,3)}\right] \propto \left[\begin{array}{ccc} j_4 & j_3 & k \\ l & j_2 & j_1 \\ j_5 & m & j_6 \end{array} \right]_A$$

Even if $\left[K_{3,3}^{(1,1)}\right]$ and $\left[K_{3,3}^{(3,3)}\right]$ look similar, we were not able to conclude that they are exactly the same; this is partly also due to the freedom in the choice of the orientation of the surface, which might have affected the results. We expect, however, that they are equal up to a sign and a factor involving the parameter A, as in the case of the complete graph on 4 vertices embedded in the torus. We might be able to solve this ambiguity by defining the evaluation of the toroidal phase factor as a sum over both possible crossings rather than just a single twist factor, i.e. we would have a factor of $A^{+2[\dots]} + A^{-2[\dots]}$ instead. This is only a suggestion which will be verify in a paper coming soon.

Finally, we consider the embedding of $K_{3,3}$ in the double torus. As mentioned before, this embedding has only one 2-cell, thus, it is not possible to reduce by means of Moussouris' algorithm. Hence, in order to decompose it, it would be necessary to change the orientation of a vertex by the "twisting" operation defined above. This would give an overall twist factor dependent on the edges involved. This operation is, however, highly arbitrary since, depending on the choice of the vertex to be twisted, one obtains either of the embeddings above or even the original embedding, thus, it is not a viable way to proceed.

Nevertheless, since we were able to identify the toroidal phase factor with the handle of the torus and obtained (up to orientation of the surface) a symbol corresponding to this surface, one might ask if all spin networks embeddable in an orientable closed surface with genus > 0 could be expressed as a sum of products of (quantum) $6j$- and toroidal symbols, one for each handle. It is not hard to imagine the existence of graphs with such evaluation. At this point the embedding of the $(3,3)$-bipartite graph in the double torus is of great interest since it might be the missing link needed to generalize the Decomposition Theorem for spin networks with genus > 1. We could use the inverse operations of the ones used in Moussouris' algorithm on this embedding in order to introduce enough vertices and edges such that the resulting graph has two components, one on each handle, which are at least 3-edge connected to each other. It would then be possible to separate the components using the generalized Wigner-Eckart theorem, [28]. This could help us to study the possibility of an evaluation of non-planar graphs as a sum of products of (quantum) $6j$- and toroidal symbols. The results of these considerations are expected in the near future.

Conclusion

In this dissertation we were able to explore a broad scope of different topics involved in the description of combinatorial manifolds in terms of spin networks. We explored briefly the possible significance of these objects for a description of space in terms of abstract objects derived from the properties of the category of spin representations and their non-classical counterpart, the quantum group $U_q(\mathfrak{sl}_2)$. Each one of the fields presented here is of interest on its own, nevertheless, together they give a description of spin networks at very different levels. For instance, the diagrammatical language of these objects encoding algebraical notions, relations and operations can be analyzed in the setting of (topological) graph theory, as well as in the context of combinatorial manifolds. This point of view helped us to identify some key aspects of the structure of spin networks, such as non-planarity and the information encoded in the graphs representing these objects.

From this, we were able to analyze the Decomposition Theorem and Moussouris' algorithm

involved in its proof in order to improve its statement which, as we noticed, was not clear enough. However, the generalization of the Decomposition Theorem for networks of higher genus is not completed. This is in part due to a generalization of Kuratowski's theorem, which needs to be considered in a thorough manner. It states the existence of a finite family of minimal forbidden subgraphs for each surface, i.e. graphs which are not embeddable in the given surface. This theorem might lower our expectations of being able to express a given graph in terms of toroidal symbols since we expect other spin networks to be non-toroidal. For instance, there are more than 800 minimal forbidden graphs known for the torus, [9]. Thus, we need to analyze the general case in order to determine if the evaluation of all spin networks with genus > 1 is expressable as a sum of products of quantum $6j$- and toroidal symbols. We expect, however, no constraint regarding the form of the evaluation of these type of networks since we could use the generalized Wigner-Eckart-theorem to divide the network in its components representing the tori of the surface.

The fact that the only graph necessary to study was $K_{3,3}$ is not surprising since it is the only 3-valent graph responsible for non-planarity and every graph of higher valence is expandable to a trivalent graph. However, what about trivalent graphs belonging to other minimal forbidden families? Again, the information used for extracting the factors needed in the evaluation is the only information contained in the network as a graph theoretical object. Thus, we do not expect a further complication due to these graphs. In any case we can be sure that the evaluation can be expressed, at least, in terms of q-$6j$-symbols and twist factors.

We saw explicitly the importance of the orientation of the vertices for the evaluation of spin networks via Moussouris' algorithm. The introduction of the twist factor in the evaluation of the spin networks destroys or complicates many of the identities between $6j$-symbols and, especially, $9j$-symbols which have, as we saw, the same form as the toroidal symbols if we disregard the twist factor. Further study of this symbol is necessary, for instance its symmetries or whether there are other relations between them similar to the ones for $9j$-symbols.

Notice that the objects needed for a possible description of quantum gravity are not spin networks themselves but rather their 4-dimensional Lorenzian analog, called spin foams, which can be seen as the time evolution of the spin networks in the underlying space. The rich structure of these objects arising from the different perspectives gives a strong argument for their study in the context of quantum gravity. For instance, an indirect related theory called Causal Dynamical Triangulation (CDT) succeeded in constructing, as an infrared limit, the 4-dimensional Minkowski spacetime, [1, 2, 3]. On the other hand, one may ask whether it is possible to use the framework presented in this dissertation, or related ones, to study the description of the Standard Model in terms of similar objects called braided ribbon networks and the emergence of matter as topological excitations of a given quantum geometry, [12, 10, 11], or even the emergence of locality and geometry itself, [22]. Furthermore, there is evidence that spin foam models could be related to gravity with a positive cosmological constant given by some relation involving the root of unity r of the parameter q of the quantum group, [15].

The above list is not, in any way, exhaustive and it is probably excluding many other interesting aspects of topics related to spin networks, however, it is merely intended as a suggestion of further reading and as an example of the rich structure behind the concepts

described throughout the dissertation.

Finally, we give a consideration related to the topics above. There is a similar theorem to Kuratowski's one for graphs embedded in three dimensional spaces. It states that there are seven distinct graphs, all containing $K_{3,3}$, which are not embeddable in 3 dimensions without a link. However, the only networks considered in the structures described, for instance, in CDT are trivial embeddings of graphs in 3-dimensional spaces; trivial in the sense that there are no links to consider. It would be interesting to study whether this extra structure gives new features useful to describe physical concepts such as, for example, matter in a theory of quantum gravity. The author is conscious of the highly speculative nature of this consideration, however, he regards as important to raise humbly the issue (in a probably very naive way) of the possible necessity to consider more basic notions, than for instance geometry, to tackle the difficulties encounter in quantum gravity.

Bibliography

[1] J. Ambjørn, J. Jurkiewicz, and R. Loll, *Reconstructing the universe*, Physical Review D **72** (2005), no. 6, 064014.

[2] ———, *The universe from scratch*, (2005).

[3] J. Ambjorn, J. Jurkiewicz, and R. Loll, *Quantum gravity, or the art of building spacetime*, Arxiv preprint hep-th/0604212 (2006).

[4] M. Atiyah, *Topological quantum field theories*, **68** (1988), no. 1, 175–186.

[5] M.F. Atiyah, *The geometry and physics of knots*, Cambridge Univ Pr, 1990.

[6] J.W. Barrett and I. Naish-Guzman, *The ponzano-regge model*, Classical and Quantum Gravity **26** (2009), 155014.

[7] J.W. Barrett and B.W. Westbury, *Invariants of piecewise-linear 3-manifolds*, Transactions of the American Mathematical Society **348** (1996), no. 10, 3997–4022.

[8] ———, *Spherical categories*, Advances in Mathematics **143** (1999), no. 2, 357–375.

[9] L.W. Beineke and R.J. Wilson, *Graph connections: Relationship between graph theory and other areas of mathematics*, Clarendon Press, 1997.

[10] S. Bilson-Thompson, J. Hackett, L. Kauffman, and L. Smolin, *Particle identifications from symmetries of braided ribbon network invariants*, Arxiv preprint arXiv:0804.0037 (2008).

[11] S. Bilson-Thompson, J. Hackett, L. Kauffman, and Y. Wan, *Emergent braided matter of quantum geometry*, Arxiv preprint arXiv:1109.0080 (2011).

[12] S.O. Bilson-Thompson, F. Markopoulou, and L. Smolin, *Quantum gravity and the standard model*, Classical and Quantum Gravity **24** (2007), 3975.

[13] J.S. Carter, D.E. Flath, and M. Saito, *The classical and quantum 6j-symbols*, vol. 43, Princeton Univ Pr, 1995.

[14] A.R. Edmonds, *Angular momentum in quantum mechanics*, Princeton Univ Pr, 1996.

[15] W.J. Fairbairn and C. Meusburger, *Quantum deformation of two four-dimensional spin foam models*, Arxiv preprint arXiv:1012.4784 (2010).

[16] N. Hartsfield and G. Ringel, *Pearls in graph theory: a comprehensive introduction*, Dover Pubns, 2003.

[17] K. Ito, *Encyclopedic dictionary of mathematics*, The MIT Pr., 1993.

[18] V.F.R. Jones, *The Jones Polynomial*, preprint (2005).

[19] L.H. Kauffman, *Knots and Physics: Series on Knots and Everything*, World Scientific Publishing Company, 2001.

[20] L.H. Kauffman and S. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, Princeton Univ Pr, 1994.

[21] G.M. Kelly and M.I. Laplaza, *Coherence for compact closed categories*, J. Pure Appl. Algebra 19 (1980), 193-213.

[22] T. Konopka, F. Markopoulou, and S. Severini, *Quantum graphity: a model of emergent locality*, Physical Review D **77** (2008), no. 10, 104029.

[23] WBR Lickorish, *Simplicial moves on complexes and manifolds*, Geometry and Topology Monographs **2** (1999), no. 299-320, 314.

[24] S. Mac Lane, *Categories for the Working Mathematician*, 1971.

[25] S. Majid, *Foundations of quantum group theory*, Cambridge Univ Pr, 2000.

[26] S.A. Major, *A spin network primer*, American Journal of Physics **67** (1999), 972.

[27] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, WH Freeman & co, 1973.

[28] J.P. Moussouris, *Quantum models of space-time based on recoupling theory*, (1983).

[29] Mikio Nakahara, *Geometry, topology and physics, second edition (graduate student series in physics)*, 2 ed., Taylor & Francis, 2003.

[30] R. Penrose, *Angular momentum: an approach to combinatorial space-time*, Quantum theory and beyond (1971), 151–180.

[31] ______, *Applications of negative dimensional tensors*, Combinatorial mathematics and its applications (1971), 221–244.

[32] S. Piunikhin, *Turaev-Viro and Kauffman-Lins invariants for 3-manifolds coincide*, J. Knot Theory Ramifications **1** (1992), 105–135.

[33] G. Ponzano and T. Regge, *SEMICLASSICAL LIMIT OF RACAH COEFFICIENTS.*, Tech. report, Princeton Univ., NJ, 1969.

[34] T. Regge, *General relativity without coordinates*, Il Nuovo Cimento (1955-1965) **19** (1961), no. 3, 558–571.

[35] C. Rovelli and L. Smolin, *Spin networks and quantum gravity*, Physical Review D **52** (1995), no. 10, 5743.

[36] S. Sawin, *Links, quantum groups and tqfts*, BULLETIN-AMERICAN MATHEMATICAL SOCIETY **33** (1996), 413–446.

[37] J. Smith, *Introduction to abstract algebra*, CRC Press, 2009.

[38] Ivan Soprunov, *Spherical geometry and euler's formula*, Spherical Geometry Class Notes.

[39] V.G. Turaev and O.Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology **31** (1992), no. 4, 865–902.

[40] O. Viro, *Moves of Triangulations of a PL-Manifold*, Quantum Groups (1992), 367–372.